A COURSE OF

MODERN ANALYSIS

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AN INTRODUCTION TO THE GENERAL THEORY OF INFINITE SERIES AND OF ANALYTIC FUNCTIONS; WITH AN ACCOUNT OF THE PRINCIPAL TRANSCENDENTAL FUNCTIONS

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PREFACE.

THE first half of this book contains an account of those methods and processes of higher mathematical analysis, which seem to be of greatest importance at the present time; as will be seen by a glance at the table of contents, it is chiefly concerned with the properties of infinite series and complex integrals, and their applications to the analytical expression of functions. A discussion of infinite determinants and of asymptotic expansions has been included, as it seemed to be called for by the value of these theories in connexion with linear differential equations and astronomy.

In the second half of the book, the methods of the earlier part are applied in order to furnish the theory of the principal functions of analysis—the Gamma, Legendre, Bessel, Hypergeometric, and Elliptic Functions. An account has also been given of those solutions of the partial differential equations of mathematical physics, which can be constructed by the help of these functions.

My grateful thanks are due to two members of Trinity College, Rev. E. M. Radford, M.A. (now of St John's School, Leatherhead), and Mr J. E. Wright, B.A., who with great kindness and care have read the proof-sheets; and to Professor Forsyth, for many helpful consultations during the progress of the work. My great indebtedness to Dr Hobson's memoirs on Legendre functions must be specially mentioned here; and I must thank the staff of the University Press for their excellent co-operation in the production of the volume.

E. T. WHITTAKER.

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PART I.

THE PROCESSES OF ANALYSIS.

CHAPTER I.

Complex Numbers.

1. *Real Numbers.*

The idea of a set of numbers is derived in the first instance from the consideration of the set of *positive integral numbers*, or *positive integers*; that is to say, the numbers 1, 2, 3, 4, ... Positive integers have many properties, which will be found in treatises on the Theory of Integral Numbers; but at a very early stage in the development of mathematics it was found that they are inadequate to express all the quantities occurring in calculations; and so this primitive number system has come to be enlarged. In elementary Arithmetic, and in the arithmetical applications of Algebra, several new classes of numbers are defined, namely *rational fractions* such as $\frac{1}{2}$, *negative numbers* such as -3, and *irrational numbers* such as the number 1·414213..., which represents the square root of 2.

The object of the introduction of these extended types of number is that we may express the result of performing the operations of addition, subtraction, multiplication, division, involution, and evolution, on all integral numbers. Thus, the result of dividing the integer 1 by the integer 2 is inexpressible until we introduce the idea of fractional numbers: and the result of subtracting the integer 2 from the integer 1 is inexpressible until we introduce the idea of negative numbers.

The totality of the numbers introduced up to this point is called the *aggregate of real numbers.*

The extension of the idea of number, which has just been described, was not effected without some opposition from the more conservative mathematicians. In the latter half of the 18th century, Maseres (1731—1824) and Frend (1757—1841) published works on Algebra, Trigonometry, etc., in which the use of negative quantities was disallowed, although Descartes had used them unrestrictedly more than a hundred years before.

2. *Complex Numbers**.

If we attempt to perform the operations already named—multiplication, etc.—on any of the real numbers thus recognised, we find that there is one case in which the result of the operation cannot be expressed without the introduction of yet another type of numbers. The case referred to is that in which the operation of evolution is applied to a negative number, e.g. to find the square root of -2. To express the results of this and similar operations, we make use of a new number, denoted by the letter i; this is defined as *a quantity which satisfies the fundamental laws of algebra* (i.e. can be combined with other numbers according to the associative, distributive, and commutative laws) *and has for its square the negative number* -1.

It is easily seen that all the quantities which can be formed by combining i with real numbers are of the form $a+bi$, where a and b are real numbers. A quantity $a+bi$ of this nature is called (after Gauss) a *complex number.* Real numbers may be regarded as a particular case of complex numbers, corresponding to a zero value of the quantity b.

The complex quantity thus introduced may in the first instance be regarded as formed by the association of the pair of real numbers a and b; as the quantities a, b, i are subject to the ordinary laws of algebra, we obtain for the addition and multiplication of two complex numbers $a+bi$ and $c+di$ the formulae

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$
$$(a+bi)(c+di) \quad =(ac-bd)+(ad+bc)i.$$

But a complex number will usually be considered apart from its composition, as an irresoluble entity. Regarded in this light, it satisfies the fundamental laws of algebra; so that if a, b, c are complex numbers, we have

$$a+b=b+a,$$
$$ab=ba,$$
$$(a+b)+c=a+(b+c),$$
$$ab\,.\,c=a\,.\,bc,$$
$$a(b+c)=ab+ac.$$

It is found that the operations of multiplication, etc., when applied to complex numbers, do not lead to numbers of any fresh type; the complex number will therefore for our purposes be taken as the most general type of number.

The introduction of the complex number has led to many important developments in mathematics. Functions which, when real variables only

* For the general theory of complex numbers, see Hankel, *Theorie der complexen Zahlensysteme* (Leipzig, 1867), and Stolz, *Vorlesungen über allgemeine Arithmetik II.* (Leipzig, 1886).

are considered, appear as essentially distinct, are seen to be connected when complex variables are introduced: thus the circular functions are found to be expressible in terms of exponential functions of a complex argument, by the equations

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}),$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Again, many of the most important theorems of modern analysis are not true if the quantities concerned are restricted to be real; thus, the theorem that every algebraic equation of degree n has n roots is true in general only when complex values of the roots are admitted.

Hamilton's quaternions furnish an example of a still further extension of the idea of number. A quaternion

$$w + xi + yj + zk$$

is formed from four real numbers w, x, y, z, and four number-units 1, i, j, k, in the same way as the ordinary complex number $x + iy$ is formed from two real numbers x, y, and two number-units 1, i. Quaternions however do not obey the commutative law of multiplication.

3. *The modulus of a complex quantity.*

Let $x + iy$ be a complex quantity; x and y being real numbers. Then the positive square root of $x^2 + y^2$ is called the *modulus* of $(x + yi)$, and is written

$$|x + yi|.$$

Let us consider the complex number which is the sum of two known complex numbers, $x + iy$ and $u + iv$. We have

$$(x + iy) + (u + iv) = (x + u) + i(y + v).$$

The modulus of the sum of the two numbers is therefore

$$\{(x + u)^2 + (y + v)^2\}^{\frac{1}{2}},$$

or

$$\{(x^2 + y^2) + (u^2 + v^2) + 2(xu + yv)\}^{\frac{1}{2}}.$$

But

$$\begin{aligned}\{|x + iy| + |u + iv|\}^2 &= \{(x^2 + y^2)^{\frac{1}{2}} + (u^2 + v^2)^{\frac{1}{2}}\}^2 \\ &= (x^2 + y^2) + (u^2 + v^2) + 2(x^2 + y^2)^{\frac{1}{2}}(u^2 + v^2)^{\frac{1}{2}} \\ &= (x^2 + y^2) + (u^2 + v^2) + 2\{(xu + yv)^2 + (xv - yu)^2\}^{\frac{1}{2}},\end{aligned}$$

and this latter expression is greater than (or at least equal to)

$$(x^2 + y^2) + (u^2 + v^2) + 2(xu + yv).$$

We have therefore

$$|x + iy| + |u + iv| \geqslant |(x + iy) + (u + iv)|,$$

or *the modulus of the sum of two complex numbers cannot be greater than the sum of their moduli*; and in general it follows that the modulus of the sum

of any number of complex quantities cannot be greater than the sum of their moduli.

Let us consider next the complex number which is the product of two known complex numbers $x+iy$ and $u+iv$; we have

$$(x+iy)(u+iv)=(xu-yv)+i(xv+yu),$$

and therefore

$$\begin{aligned}|(x+iy)(u+iv)| &= \{(xu-yv)^2+(xv+yu)^2\}^{\frac{1}{2}}\\ &= \{(x^2+y^2)(u^2+v^2)\}^{\frac{1}{2}}\\ &= |x+iy|\,|u+iv|.\end{aligned}$$

The modulus of the product of two complex quantities (and hence of any number of complex quantities) *is therefore equal to the product of their moduli.*

4. *The geometrical interpretation of complex numbers.*

For many purposes it is useful to represent complex numbers by a geometrical diagram, which may be done in the following way.

Take rectangular axes Ox, Oy, in a plane. Then a point P whose coordinates referred to these axes are x, y, will be regarded as representing the complex number $x+iy$. In this way, to every point of the plane there corresponds some complex number; and conversely, to every possible complex number there corresponds one and only one point of the plane.

The complex number $x+iy$ may be denoted by a single letter z. The point P is then called the *representative point* or *affix* of the value z; we shall also speak of the number z as being the *affix* of the point P.

If we denote $(x^2+y^2)^{\frac{1}{2}}$ by r and $\tan^{-1}\left(\frac{y}{x}\right)$ by θ, then r and θ are clearly the radius vector and vectorial angle of the point P, referred to the origin O and axis Ox.

The representation of complex quantities thus afforded is often called the *Argand diagram**.

If P_1 and P_2 are the representative points corresponding to values z_1 and z_2 respectively of z, then the point which represents the value z_1+z_2 is clearly the terminus of a line drawn from P_1, equal and parallel to that which joins the origin to P_2.

To find the point which represents the complex number z_1z_2, where z_1 and z_2 are two given complex numbers, we notice that if

$$z_1=r_1(\cos\theta_1+i\sin\theta_1),$$

$$z_2=r_2(\cos\theta_2+i\sin\theta_2),$$

* J. R. Argand published it in 1806; it had however previously been used by Gauss, and by Caspar Wessel, who discussed it in a memoir published in 1797 to the Danish Academy.

then by multiplication

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}.$$

The point which represents the value $z_1 z_2$ has therefore a radius vector measured by the product of the radii vectores of P_1 and P_2, and a vectorial angle equal to the sum of the vectorial angles of P_1 and P_2.

Miscellaneous Examples.

1. Shew that the representative points of the complex numbers $1+4i$, $2+7i$, $3+10i$, are collinear.

2. Shew that a parabola can be drawn to pass through the representative points of the complex numbers

$$2+i, \quad 4+4i, \quad 6+9i, \quad 8+16i, \quad 10+25i.$$

3. Determine by aid of the Argand diagram the nth roots of unity; and shew that the number of primitive roots (roots the powers of each of which give all the roots) is the number of integers including unity less than n and prime to it.

Prove that if $\theta_1, \theta_2, \theta_3, \ldots$ be the arguments of the primitive roots, $\Sigma \cos p\theta = 0$ when p is a positive integer less than $\dfrac{n}{abc \ldots k}$, where $a, b, c, \ldots k$ are the different constituent primes of n; and that, when $p = \dfrac{n}{abc \ldots k}$, $\Sigma \cos p\theta = \dfrac{(-1)^\mu n}{abc \ldots k}$, where μ is the number of the constituent primes.

(Cambridge Mathematical Tripos, Part I. 1895.)

CHAPTER II.

The Theory of Absolute Convergence.

5. *The limit of a sequence of quantities.*

Let $z_1, z_2, z_3, \ldots$ be a sequence of quantities (real or complex), infinite in number. The sequence is said to tend towards a *limiting value* or *limit* l, provided that, corresponding to every positive quantity ϵ, however small, a number n can be chosen, such that the inequality

$$|z_m - l| < \epsilon$$

is true for all values of m greater than n. If z is a variable quantity which takes in succession the values $z_1, z_2, z_3, \ldots$, then z is said to *tend to the limit* l.

Example. Consider the sequence of numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, for which $z_n = \frac{1}{2^n}$. This sequence tends to the limiting value $l=0$; for if any positive quantity ϵ be taken, and if n denote the integer next greater than $-\frac{\log \epsilon}{\log 2}$, then the inequality

$$\frac{1}{2^m} < \epsilon$$

is true for all values of m greater than n.

6. *The necessary and sufficient condition for the existence of a limit.*

We shall now shew that the necessary and sufficient condition for the existence of a limiting value of a sequence of finite numbers $z_1, z_2, z_3, \ldots$ is that *corresponding to any given positive quantity* ϵ, *however small, it shall be possible to find a number* n *such that the equation*

$$|z_{n+p} - z_n| < \epsilon$$

is verified for all positive integral values of p. This may be expressed in words by the statement that *a finite variable quantity has a limit if, and only if, its oscillations have the limit zero*; it may be regarded as one of the fundamental theorems of analysis.

First, we have to shew that this condition is *necessary*, i.e. that it is satisfied whenever a limit exists. Suppose then that a limit l exists; then

(§ 5) corresponding to any positive quantity ϵ, however small, a number n can be chosen such that

$$|z_n - l| < \frac{\epsilon}{2},$$

and
$$|z_{n+p} - l| < \frac{\epsilon}{2}, \text{ for all values of } p;$$

therefore

$$\begin{aligned} |z_{n+p} - z_n| &< |(z_{n+p} - l) - (z_n - l)| \\ &< |z_{n+p} - l| + |z_n - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

which shews the *necessity* of the condition

$$|z_{n+p} - z_n| < \epsilon,$$

and thus establishes the first half of the theorem.

Secondly, we have to shew that this condition is *sufficient*, i.e. that if it is satisfied, then a limit exists. Suppose then that this condition is satisfied. Let

$$z_r = x_r + iy_r,$$

where x_r and iy_r are the real and imaginary parts of z_r. Then if

$$|z_{n+p} - z_n| < \epsilon,$$

we have
$$|(x_{n+p} - x_n) + i(y_{n+p} - y_n)| < \epsilon,$$

or
$$(x_{n+p} - x_n)^2 + (y_{n+p} - y_n)^2 < \epsilon^2,$$

and therefore
$$x_n - \epsilon < x_{n+p} < x_n + \epsilon,$$

and
$$y_n - \epsilon < y_{n+p} < y_n + \epsilon.$$

Now the number n is determined by the quantity ϵ, which can be assigned arbitrarily. Let $n_1, n_2, n_3, n_4, \dots$ be the numbers which correspond in this way to the quantities $\frac{\epsilon}{2}, \frac{\epsilon}{4}, \frac{\epsilon}{8}, \frac{\epsilon}{16}, \dots$. Let u_k be the least of the quantities $x_n + \epsilon,\ x_{n_1} + \frac{\epsilon}{2},\ x_{n_2} + \frac{\epsilon}{4}, \dots x_{n_k} + \frac{\epsilon}{2^k}$, so that the quantities $u_0, u_1, u_2, \dots$ are a decreasing sequence; and let v_k be the greatest of the quantities

$$x_n - \epsilon, \quad x_{n_1} - \frac{\epsilon}{2}, \quad x_{n_2} - \frac{\epsilon}{4}, \dots x_{n_k} - \frac{\epsilon}{2^k},$$

so that the quantities $v_0, v_1, v_2, \dots$ are an increasing sequence; and clearly

$$u_k - v_k < \frac{\epsilon}{2^{k-1}}.$$

Then any of the numbers in the u-sequence is greater than any of the numbers in the v-sequence, since we have

$$u_r > v_r > v_s, \text{ if } r > s,$$

and
$$u_r > u_s > v_s, \text{ if } r < s;$$

and the difference $u_k - v_k$ can be made as small as we please by increasing k. These two sequences u and v therefore uniquely define a real number (rational or irrational) ξ, such that ξ is less than any number in the u-sequence and greater than any number in the v-sequence, and the differences $u_k - \xi$ and $\xi - v_k$ can be made as small as we please by increasing k.

Then $$u_k - \xi < u_k - v_k < \frac{\epsilon}{2^{k-1}},$$

so $$|x_{n_k} - \xi| < |x_{n_k} - u_k| + |u_k - \xi| < \frac{\epsilon}{2^{k-1}} + \frac{\epsilon}{2^{k-1}} < \frac{\epsilon}{2^{k-2}}.$$

Moreover, by hypothesis,

$$|x_{n_k+p} - x_{n_k}| < \frac{\epsilon}{2^k},$$

where p is any positive integer; and so

$$|x_{n_k+p} - \xi| < \frac{\epsilon}{2^{k-3}}.$$

Since $\frac{\epsilon}{2^{k-3}}$ can be made as small as we wish by increasing k, this inequality shews that the sequence $x_1, x_2, x_3, \ldots$ tends to the limit ξ. Similarly the sequence $y_1, y_2, y_3, \ldots$ tends to a limit η.

Thus if τ be any small positive quantity, it is possible to choose a number m such that for all values of r greater than m we have

$$|x_r - \xi| < \frac{\tau}{\sqrt{2}}, \quad \text{and} \quad |y_r - \eta| < \frac{\tau}{\sqrt{2}},$$

and therefore $$(x_r - \xi)^2 + (y_r - \xi)^2 < \tau^2,$$

or $$|z_r - l| < \tau,$$

where $$l = \xi + i\eta.$$

This inequality shews that the sequence of quantities $z_1, z_2, z_3, \ldots$ tends to the limit l; which establishes the required result, namely that the condition expressed is *sufficient* to ensure the existence of a limit.

7. *Convergence of an infinite series.*

Let $u_1, u_2, u_3, \ldots u_n$ be a series of numbers (real or complex). Let the sum

$$u_1 + u_2 + \ldots + u_n$$

be denoted by S_n.

Then the infinite series

$$u_1 + u_2 + u_3 + u_4 + \ldots$$

is said to *be convergent*, or to *converge to a sum* S, if the sequence of numbers $S_1, S_2, S_3, \ldots$ tends to a definite limit S as n tends to infinity. In other cases, the infinite series is said to be *divergent*. When the series converges the quantity $S - S_n$, which is the sum of the series

$$u_{n+1} + u_{n+2} + u_{n+3} + \ldots,$$

is called the *remainder after n terms*, and is frequently denoted by the symbol R_n.

The sum $$u_{n+1} + u_{n+2} + \dots + u_{n+p}$$
will be denoted by $S_{n,p}$.

It follows at once, by combining the above definition with the results of the last paragraph, that the necessary and sufficient condition for the convergence of an infinite series is that $S_{n,p}$ shall tend to the limit zero as n tends to infinity, whatever p is.

Since $u_{n+1} = S_{n,1}$, it follows as a particular case that u_{n+1} must tend to zero as n tends to infinity,—in other words, the terms of a convergent series must ultimately become indefinitely small. But this last condition, though necessary, is not sufficient in itself to ensure the convergence of the series, as appears from a study of the series
$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \dots.$$
In this series,
$$S_{n,n} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n},$$
so
$$S_{n,n} > \frac{n}{2n},$$
or
$$S_{n,n} > \tfrac{1}{2}.$$
Therefore
$$\begin{aligned} S &= 1 + S_{1,1} + S_{2,2} + S_{4,4} + S_{8,8} + S_{16,16} + \dots \\ &> 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \dots, \end{aligned}$$
which is clearly infinite; the series is therefore divergent.

Infinite series were used by Lord Brouncker in *Phil. Trans.* 1668, and the expressions convergent and divergent were introduced by Gregory in the same year. But the great mathematicians of the 18th century used infinite series freely without, for the most part, considering the question of their convergence. Thus Euler gave the sum of the series
$$\dots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \dots \quad \text{.........................} (a)$$
as zero, on the ground that
$$z + z^2 + z^3 + \dots = \frac{z}{1-z} \quad \text{.....................................} (b)$$
and
$$1 + \frac{1}{z} + \frac{1}{z^2} + \dots = \frac{z}{z-1} \quad \text{.....................................} (c).$$
The error of course arises from the fact that the series (b) converges only when $|z| < 1$, and the series (c) converges only when $|z| > 1$, so the series (a) does not converge for any value of z.

The modern theory of convergence may be said to date from the publication of Gauss' *Disquisitiones circa seriem infinitam* $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \dots$ in 1812, and Cauchy's *Analyse Algébrique* in 1821. See Reiff, *Geschichte der unendlichen Reihen* (Tübingen, 1889).

8. *Absolute convergence and semi-convergence.*

In order that the series

$$u_1 + u_2 + u_3 + u_4 + \dots$$

(which we shall frequently denote by Σu_n), whose terms are supposed to be any complex quantities, may be convergent, it is sufficient, but not necessary, that the series $\Sigma |u_n|$ shall be convergent.

For we have

$$|S_{n,p}| = |u_{n+1} + u_{n+2} + \dots + u_{n+p}|$$
$$\leqslant |u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}|,$$

and this last expression is infinitely small, whatever p may be, when n is infinitely great, provided the series $\Sigma |u_n|$ is convergent.

Although this condition is sufficient to ensure the convergence of the series Σu_n, it is not necessary, i.e. the series Σu_n can converge even when the series $\Sigma |u_n|$ diverges. This may be seen by considering the series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots.$$

This series is convergent; for writing it in the form

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots,$$

or

$$\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \dots,$$

we see that its sum is greater than $\frac{1}{2}$, and that the partial sum obtained by truncating the series after its $2n$th term increases as n increases; on the other hand, by writing it in the form

$$1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) + \dots,$$

or

$$1 - \frac{1}{6} - \frac{1}{20} - \dots,$$

we see that the sum is less than 1, and that the partial sum obtained by truncating the series after its $(2n+1)$th term decreases as n increases. These partial sums must therefore tend to some limit between $\frac{1}{2}$ and 1, and so the series converges. But the series of moduli is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which as already shewn is divergent. In this case therefore, the divergence of the series of moduli does not entail the divergence of the series itself.

Series whose convergence is due to the convergence of the series formed by the moduli of their terms possess special properties of great importance, and are called *absolutely convergent* series. Series which though convergent are not absolutely convergent (i.e. the series themselves converge, but the series of moduli diverge) are said to be *semi-convergent* or *conditionally convergent*.

9. *The geometrical series, and the series* $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

The convergence of a particular series is in most cases investigated, not by the direct consideration of the sum $S_{n,p}$, but (as will appear from the following articles) by a comparison of the given series with some other series which is known to be convergent or divergent. We shall now investigate the convergence of two of the series which are most frequently used as standards for comparison.

(1) *The geometrical series.*

The geometrical series is defined to be the series

$$1 + z + z^2 + z^3 + z^4 \dots.$$

Considering the series of moduli

$$1 + |z| + |z|^2 + |z|^3 + \dots,$$

we have for it
$$S_{n,p} = |z|^{n+1} + |z|^{n+2} + \dots + |z|^{n+p},$$

or
$$S_{n,p} = |z|^{n+1} \frac{1 - |z|^p}{1 - |z|}.$$

Now if $|z| < 1$, then $\frac{1 - |z|^p}{1 - |z|}$ is finite for all values of p, while $|z|^{n+1}$ tends to zero as n tends to infinity. The series

$$1 + |z| + |z|^2 + \dots$$

is therefore convergent so long as $|z| < 1$, and therefore *the geometric series is absolutely convergent so long as* $|z| < 1$.

When $|z| \geqslant 1$, the terms of the geometric series do not tend to zero as n increases, and the series is therefore divergent.

(2) *The series* $\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$.

Consider now the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, where s is any positive real quantity.

We have
$$\frac{1}{2^s} + \frac{1}{3^s} < \frac{2}{2^s} < \frac{1}{2^{s-1}},$$

$$\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} < \frac{4}{4^s} < \frac{1}{4^{s-1}},$$

and so on. Thus the sum of any number of terms of the series is less than the sum of the corresponding terms of the series

$$\frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \frac{1}{8^{s-1}},$$

or
$$\frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{2^{2(s-1)}} + \frac{1}{2^{3(s-1)}} + \dots,$$

and hence the convergence of this last series would involve that of the original series. But this last series is a geometrical series, and is therefore convergent if

$$\frac{1}{2^{s-1}} < 1,$$

that is, if $$s > 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *is therefore convergent if* $s > 1$; and since its terms are all real and positive, they are equal to their own moduli, and so the series of moduli of the terms is convergent; that is, *the convergence is absolute.*

If $s = 1$, the series becomes

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which we have already shewn to be divergent; and when $s \leqq 1$, it is *à fortiori* divergent, since the effect of diminishing s is to increase the terms of the series. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *is therefore divergent if* $s \leqslant 1$.

10. *The Comparison-Theorem.*

We shall now shew that *a series*

$$u_1 + u_2 + u_3 + u_4 + \dots$$

will be absolutely convergent, provided $|u_n|$ *is always less than* $C|v_n|$, *where* C *is any finite number independent of* n, *and* v_n *is the nth term of another series which is known to be absolutely convergent.*

For we have under these conditions

$$|u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}| < C\{|v_{n+1}| + |v_{n+2}| + \dots + |v_{n+p}|\},$$

where n and p are any integers. But since the series Σv_n is absolutely convergent, the series $\Sigma |v_n|$ is convergent, and so

$$|v_{n+1}| + |v_{n+2}| + \dots + |v_{n+p}|$$

tends to zero as n increases, whatever p may be. It follows therefore that

$$|u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}|$$

tends to zero as n increases, whatever p may be, i.e. the series $\Sigma |u_n|$ is convergent. The series Σu_n is therefore absolutely convergent.

Corollary. A series will be absolutely convergent if the ratio of its terms, to the corresponding terms of a series which is known to be absolutely convergent, is always finite.

Example 1. Shew that the series

$$\cos z + \frac{1}{2^2}\cos 2z + \frac{1}{3^2}\cos 3z + \frac{1}{4^2}\cos 4z + \dots$$

is absolutely convergent for all real values of z.

For when z is real, we have $|\cos nz| \leqslant 1$, and therefore $\left|\frac{\cos nz}{n^2}\right| \leqslant \frac{1}{n^2}$. The moduli of the terms of the given series are therefore less than, or at most equal to, the corresponding terms of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots,$$

which by § 9 is absolutely convergent. The given series is therefore absolutely convergent.

Example 2. Shew that the series

$$\frac{1}{1^2(z-z_1)}+\frac{1}{2^2(z-z_2)}+\frac{1}{3^2(z-z_3)}+\frac{1}{4^2(z-z_4)}+\ldots,$$

where $$z_n=\left(1+\frac{1}{n}\right)e^{ni}, \qquad (n=1, 2, 3, \ldots)$$

is convergent for all values of z, except the values $z=z_1, z_2, z_3, \ldots$.

The geometric representation of complex numbers is helpful in discussing a question of this kind. Let values of the complex number z be represented on a plane: then the values $z_1, z_2, z_3, \ldots$ will form a series of points which for large values of n lie very near the circumference of the circle whose centre is the origin and whose radius is unity: so that in fact the whole circumference of this circle may be regarded as composed of points included in the values z_n.

For these special values z_n of z, the given series is clearly divergent, since the term $\frac{1}{n^2(z-z_n)}$ becomes infinite when $z=z_n$. The series is therefore divergent at all points z situated on the circumference of the circle of radius unity.

Suppose now that z has a value which is distinct from any of the values z_n. Then $\frac{1}{|z-z_n|}$ is finite for all values of n, and less than some definite upper limit c: so the moduli of the terms of the given series are less than the corresponding terms of the series

$$\frac{c}{1^2}+\frac{c}{2^2}+\frac{c}{3^2}+\frac{c}{4^2}+\ldots,$$

which is known to be absolutely convergent. The given series is therefore absolutely convergent for all values of z, except the values z_n.

It is interesting to notice that the area in the z-plane over which the series converges is divided into two parts, between which there is no intercommunication, by the circle $|z|=1$.

Example 3. Shew that the series

$$2\sin\frac{z}{3}+4\sin\frac{z}{9}+8\sin\frac{z}{27}+\ldots+2^n\sin\frac{z}{3^n}+\ldots$$

converges absolutely for all finite values of z.

For when n is large, the quantity

$$\frac{\left|2^n\sin\frac{z}{3^n}\right|}{\frac{2^n|z|}{3^n}}$$

has a value nearly unity; the given series is therefore absolutely convergent, since the comparison series $\Sigma\frac{2^n|z|}{3^n}$ is absolutely convergent.

11. *Discussion of a special series of importance.*

The theorem of § 10 enables us to establish the absolute convergence of a series which will be found to be of great importance in the theory of Elliptic Functions.

Let ω_1 and ω_2 be any constants whose ratio is not purely real; and consider the series

$$\frac{1}{z^2}+\Sigma\left\{\frac{1}{(z-2m\omega_1-2n\omega_2)^2}-\frac{1}{(2m\omega_1+2n\omega_2)^2}\right\},$$

where the summation extends over all positive and negative integral and zero values of m and n (the simultaneous zero values $m=0$, $n=0$ excepted). At each of the points $z=2m\omega_1+2n\omega_2$ one term of the series is infinite, and the series therefore is not convergent. The absolute convergence of the series for all other values of z can be established as follows.

Let z have any value not included in this set of exceptional values.

The series may be written

$$\frac{1}{z^2}+\Sigma\frac{1}{(2m\omega_1+2n\omega_2)^2}\left\{\left(1-\frac{z}{2m\omega_1+2n\omega_2}\right)^{-2}-1\right\}.$$

Now when $|2m\omega_1+2n\omega_2|$ is large (and we can suppose the series arranged in order of magnitude of $|2m\omega_1+2n\omega_2|$), we have

$$\text{Limit}\,\frac{\left(1-\dfrac{z}{2m\omega_1+2n\omega_2}\right)^{-2}-1}{\dfrac{2z}{2m\omega_1+2n\omega_2}}=1.$$

The series is therefore absolutely convergent if the series

$$\Sigma\frac{2z}{(2m\omega_1+2n\omega_2)^3}$$

is absolutely convergent: that is, if the series

$$\Sigma\frac{1}{(2m\omega_1+2n\omega_2)^3}$$

is absolutely convergent.

To discuss the convergence of the latter series, let

$$\omega_1=\alpha_1+i\beta_1,\quad \omega_2=\alpha_2+i\beta_2,$$

where α_1, α_2, β_1, β_2, are real. Then the series of moduli of the terms of this series is

$$\Sigma\frac{1}{\{(\alpha_1m+\alpha_2n)^2+(\beta_1m+\beta_2n)^2\}^{\frac{3}{2}}}.$$

This converges if the series

$$\Sigma\frac{1}{(m^2+n^2)^{\frac{3}{2}}}\qquad\text{(which we may denote by } S)$$

converges; for the quotient of corresponding terms is

$$\left\{\frac{(\alpha_1+\alpha_2\mu)^2+(\beta_1+\beta_2\mu)^2}{1+\mu^2}\right\}^{\frac{3}{2}},$$

where

$$\mu=\frac{n}{m};$$

and this is never zero or infinite.

We have therefore only to study the convergence of the series S. Now

$$S=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m^2+n^2)^{\frac{3}{2}}}$$

$$=4\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m^2+n^2)^{\frac{3}{2}}},$$

where in the summation the occurrence of the pair of values $m=0$, $n=0$ together is excluded.

Separating S into the terms for which $m=n$, $m>n$, and $m<n$, respectively, we have

$$\tfrac{1}{4}S=\sum_{m=1}^{\infty}\frac{1}{(2m^2)^{\frac{3}{2}}}+\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\frac{1}{(m^2+n^2)^{\frac{3}{2}}}+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{1}{(m^2+n^2)^{\frac{3}{2}}}.$$

But

$$\sum_{n=0}^{m-1}\frac{1}{(m^2+n^2)^{\frac{3}{2}}}<\frac{m}{(m^2)^{\frac{3}{2}}}<\frac{1}{m^2}.$$

Therefore

$$\tfrac{1}{4}S<\sum_{m=1}^{\infty}\frac{1}{2^{\frac{3}{2}}m^3}+\sum_{m=1}^{\infty}\frac{1}{m^2}+\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

But the series $\sum_{m=1}^{\infty}\frac{1}{m^3}$ and $\sum_{m=1}^{\infty}\frac{1}{m^2}$ are known to be convergent. So the series S is absolutely convergent. The original series is therefore absolutely convergent for all values of z except the specified excluded values.

Example. Prove that the series

$$\Sigma\frac{1}{(m_1^2+m_2^2+\ldots+m_r^2)^{\mu}},$$

in which the summation extends over all positive and negative integral values and zero values of $m_1, m_2, \ldots m_r$, except the set of simultaneous zero values, is absolutely convergent if $\mu>\frac{r}{2}$. (Eisenstein, *Crelle's Journal*, XXXV.)

12. *A convergency-test which depends on the ratio of the successive terms of a series.*

We shall now shew that *a series*

$$u_1+u_2+u_3+u_4+\ldots$$

is absolutely convergent, provided that for all values of n greater than some

fixed value r, the quantity $\left|\frac{u_{n+1}}{u_n}\right|$ *is less than K, where K is some positive quantity independent of n and less than unity.*

For the terms of the series

$$|u_{r+1}| + |u_{r+2}| + |u_{r+3}| + \dots$$

are respectively less than the terms of the series

$$|u_{r+1}|(1 + K + K^2 + K^3 + \dots),$$

which is a geometric series, and therefore absolutely convergent when $K < 1$.

Thus if $\left|\frac{u_{n+1}}{u_n}\right|$ tends as n increases to a limiting value which is less than unity, the series is absolutely convergent.

Example 1. If $|c| < 1$, shew that the series

$$\sum_{n=1}^{\infty} c^{n^2} e^{nz}$$

converges absolutely for all values of z.

For the ratio of the $(n+1)$th term to the nth is

$$c^{(n+1)^2 - n^2} e^z,$$

or

$$c^{2n+1} e^z,$$

and if $|c| < 1$, this is ultimately indefinitely small.

Example 2. Shew that the series

$$z + \frac{a-b}{2!} z^2 + \frac{(a-b)(a-2b)}{3!} z^3 + \frac{(a-b)(a-2b)(a-3b)}{4!} z^4 + \dots$$

converges absolutely so long as $|z| < \frac{1}{|b|}$.

For the ratio of the $(n+1)$th term to the nth is $\frac{a-nb}{n+1} z$, or ultimately $-bz$: so the condition for absolute convergence is $|bz| < 1$, or $|z| < \frac{1}{|b|}$.

Example 3. Shew that the series $\sum_{n=1}^{\infty} \frac{nz^{n-1}}{z^n - \left(1 + \frac{1}{n}\right)^n}$ converges absolutely so long as $|z| < 1$.

For when $|z| < 1$, the terms of the series bear a finite ratio to those of the series $\sum_{n=1}^{\infty} nz^{n-1}$; but this latter series is then absolutely convergent, since the ratio of the $(n+1)$th term to the nth is $\left(1 + \frac{1}{n}\right) z$, which tends to a limit less than unity as n increases.

13. *A general theorem on series for which* $\underset{n=\infty}{\text{Limit}} \left|\frac{u_{n+1}}{u_n}\right| = 1$.

It is obvious that if, for all values of n greater than some fixed value r,

$|u_{n+1}|$ is greater than $|u_n|$, then the terms of the series do not tend to zero as n increases, and the series is therefore divergent. On the other hand, if $\left|\frac{u_{n+1}}{u_n}\right|$ is always less than some quantity which is itself less than unity, we have shewn in § 12 that the series is absolutely convergent. The limiting case is that in which, as n increases, $\left|\frac{u_{n+1}}{u_n}\right|$ tends to the value unity. In this case a further investigation is necessary.

We shall now shew that *a series*

$$u_1 + u_2 + u_3 + \ldots,$$

in which $\left|\frac{u_{n+1}}{u_n}\right|$ *tends to the limit unity as* n *increases, will be absolutely convergent if, for all values of* n *after some fixed value, we have*

$$\left|\frac{u_{n+1}}{u_n}\right| \leqslant 1 - \frac{1+c}{n},$$

where c *is a positive quantity independent of* n.

For compare the series $\Sigma\,|u_n|$ with the convergent series Σv_n, where

$$v_n = \frac{A}{n^{1+\frac{c}{2}}}$$

and A is a constant; we have

$$\frac{v_{n+1}}{v_n} = \left(\frac{n}{n+1}\right)^{1+\frac{c}{2}} = \left(1+\frac{1}{n}\right)^{-\left(1+\frac{c}{2}\right)}$$

$$= 1 - \frac{1+\frac{c}{2}}{n} + \text{terms in } \frac{1}{n^2},\ \frac{1}{n^3},\ \ldots\ldots$$

As n increases, $\frac{v_{n+1}}{v_n}$ will therefore tend to the limit

$$1 - \frac{1+\frac{c}{2}}{n}:$$

so that after some value of n we shall have

$$\left|\frac{u_{n+1}}{u_n}\right| < \frac{v_{n+1}}{v_n}.$$

By a suitable choice of the constant A, we can therefore secure that for all values of n we shall have

$$|u_n| < v_n.$$

As Σv_n is convergent, $\Sigma\,|u_n|$ is therefore convergent, and so Σu_n is absolutely convergent.

Corollary. If $\left|\dfrac{u_{n+1}}{u_n}\right|$ can be expanded in descending powers of n in the form

$$1+\frac{A_1}{n}+\frac{A_2}{n^2}+\frac{A_3}{n^3}+\dots,$$

where A_1, A_2, A_3, ... are independent of n, then the series is absolutely convergent if $A_1 < -1$.

This is easily seen to follow from the fact that when n is large the terms

$$\frac{A_2}{n^2}+\frac{A_3}{n^3}+\dots$$

become unimportant in comparison with A_1.

14. *Convergence of the hypergeometric series.*

The theorems which have been given may be illustrated by a discussion of the convergence of the *hypergeometric series*,

$$1+\frac{a\,.\,b}{1\,.\,c}z+\frac{a(a+1)\,b(b+1)}{1\,.\,2\,.\,c(c+1)}z^2+\frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1\,.\,2\,.\,3\,.\,c(c+1)(c+2)}z^3+\dots,$$

which is generally denoted by $F(a, b, c, z)$.

If c is a negative integer, all the terms after the $(1-c)$th will be infinite; and if either a or b is a negative integer the series will terminate at the $(1-a)$th or $(1-b)$th term as the case may be. We shall suppose these cases set aside, so that a, b, and c are assumed not to be negative integers.

The ratio of the $(n+1)$th term to the nth is

$$\frac{u_{n+1}}{u_n}=\frac{(a+n-1)(b+n-1)}{n(c+n-1)}z.$$

Therefore

$$\left|\frac{u_{n+1}}{u_n}\right|=\frac{\left|1+\dfrac{a-1}{n}\right|\left|1+\dfrac{b-1}{n}\right|}{\left|1+\dfrac{c-1}{n}\right|}\,|z|.$$

As n tends to infinity, this tends to the limit $|z|$. We see therefore by § 12 that *the series is absolutely convergent when* $|z|<1$, *and divergent when* $|z|>1$.

When $|z|=1$, we have

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|1+\frac{a-1}{n}\right|\left|1+\frac{b-1}{n}\right|\left|1-\frac{c-1}{n}+\frac{(c-1)^2}{n^2}-\dots\right|$$

$$=\left|1+\frac{a+b-c-1}{n}+\text{terms in }\frac{1}{n^2},\ \frac{1}{n^3},\text{ etc.}\right|.$$

Now a, b, c are in the most general case supposed to be complex numbers.

Let them be given in terms of their real and imaginary parts by the equations

$$a = a' + ia'',$$
$$b = b' + ib'',$$
$$c = c' + ic''.$$

Then (neglecting the terms in $\frac{1}{n^2}, \frac{1}{n^3}$, etc.) we have

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|1 + \frac{a' + b' - c' - 1 + i(a'' + b'' - c'')}{n}\right|$$
$$= \left\{\left(1 + \frac{a' + b' - c' - 1}{n}\right)^2 + \left(\frac{a'' + b'' - c''}{n}\right)^2\right\}^{\frac{1}{2}}$$
$$= 1 + \frac{a' + b' - c' - 1}{n} + \text{terms in } \frac{1}{n^2}, \frac{1}{n^3}, \text{ etc.}$$

By § 13, the condition for absolute convergence is

$$a' + b' - c' < 0.$$

Hence *when* $|z| = 1$, *the condition for the absolute convergence of the hypergeometric series is that the real part of* $a + b - c$ *shall be negative.*

15. *Effect of changing the order of the terms in a series.*

In an ordinary sum the order of the terms is of no importance, and can be varied without affecting the result of the addition. In an infinite series however this is no longer the case, as will appear from the following example.

Let
$$\Sigma = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \dots,$$
and
$$S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots,$$
and let Σ_n and S_n denote the sums of their first n terms. These infinite series are formed of the same terms, but the order of the terms is different.

Then if k be any positive integer,

$$\Sigma_{3k} - S_{4k} = \frac{1}{2}\left(\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{k+k}\right) = \frac{1}{2}\, p_k \text{ say.}$$

But
$$p_k - p_{k-1} = \frac{1}{2k-1} + \frac{1}{2k} - \frac{1}{k} = \frac{1}{2k-1} - \frac{1}{2k}.$$

Similarly
$$p_{k-1} - p_{k-2} = \frac{1}{2k-3} - \frac{1}{2k-2}.$$

A series of equations like this can be formed, of which the last is

$$p_1 = 1 - \tfrac{1}{2}.$$

Adding these, we have

$$p_k = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots - \frac{1}{2k} = S_{2k}.$$

Thus $$\Sigma_{3k} = S_{4k} + \frac{1}{2} S_{2k}.$$

Making k indefinitely great, this gives
$$\Sigma = \frac{3}{2} S,$$
an equation which shews that the effect of deranging the order of the terms in S has been an alteration in the value of its sum.

Example. If in the series
$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots$$
the order of the terms be altered, so that the ratio of the number of positive terms to the number of negative terms in S_n is ultimately a^2, shew that the sum of the series will become $\log (2a)$.

(Manning.)

16. *The fundamental property of absolutely convergent series.*

We shall now shew that *the sum of an absolutely convergent series is not affected by changing in any manner the order in which the terms occur.*

For let $$S = u_1 + u_2 + u_3 + u_4 + \dots$$
be an absolutely convergent series, and let S' be a series formed by the same terms in a different order.

Suppose that in order to include the first n terms of S, it is necessary to take m terms of S'. So if k be any number greater than m, we have
$$S_k' = S_n + \text{terms of } S \text{ whose suffix is greater than } n.$$
Therefore
$$|S_k' - S| \leqslant |S_n - S| + \text{the sum of the moduli of a number of terms of } S \text{ whose suffix is greater than } n$$
$$\leqslant |S_n - S| + |u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \dots .$$
When n tends to infinity, $|S_n - S|$ tends to zero since the series S is convergent, and the sum
$$|u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \dots$$
tends to zero also, since the series is absolutely convergent.

Thus $|S_k' - S|$ tends to zero when k is indefinitely increased; which establishes the required result.

17. *Riemann's theorem on semi-convergent series.*

We shall now shew that *a semi-convergent series*
$$u_1 + u_2 + u_3 + u_4 + \dots,$$
with real terms, may be made to converge to any desired real value, by suitably disposing the order in which the terms occur. This property stands in sharp contradiction to that proved in the last article; an example of it was afforded by the result of § 15.

To establish the theorem, let the positive terms in the series be

$$u_{p_1}, \quad u_{p_2}, \quad u_{p_3}, \ldots,$$

and let the negative terms be

$$u_{n_1}, \quad u_{n_2}, \quad u_{n_3}, \ldots.$$

Then the series

$$u_{p_1} + u_{p_2} + u_{p_3} + \ldots$$

and

$$-u_{n_1} - u_{n_2} - u_{n_3} - \ldots$$

cannot be both convergent: for if they were, the original series would be absolutely convergent: one of them must therefore be divergent: and the other cannot be convergent, since in that case the original series would be divergent. It follows that the series

$$u_{p_1} + u_{p_2} + u_{p_3} + \ldots$$

and

$$-u_{n_1} - u_{n_2} - u_{n_3} - \ldots$$

are both divergent.

Now let S be any real number, and let it be desired to change the order of the terms in the original series, in such a way as to cause it to converge to the sum S. Suppose that a terms of the series

$$u_{p_1} + u_{p_2} + u_{p_3} + \ldots$$

have to be taken in order to obtain a sum greater than S, so that

$$u_{p_1} + u_{p_2} + \ldots + u_{p_{a-1}} < S < u_{p_1} + u_{p_2} + \ldots + u_{p_a}.$$

Take now a number b of the terms of the series

$$u_{n_1} + u_{n_2} + u_{n_3} + \ldots,$$

such as are required to make the sum

$$u_{p_1} + u_{p_2} + \ldots + u_{p_a} + u_{n_1} + u_{n_2} + \ldots + u_{n_b}$$

less than S: so that

$$u_{p_1} + u_{p_2} + \ldots + u_{p_a} + u_{n_1} + \ldots + u_{n_{b-1}} > S > u_{p_1} + u_{p_2} + \ldots + u_{p_a} + u_{n_1} + \ldots + u_{n_b}.$$

Take next a number c of the terms of the series

$$u_{p_1} + u_{p_2} + \ldots,$$

such as are required to make the sum

$$u_{p_1} + u_{p_2} + \ldots + u_{p_a} + u_{n_1} + u_{n_2} + \ldots + u_{n_b} + u_{p_{a+1}} + \ldots + u_{p_{a+c}}$$

greater than S; and then take a number d of the terms of the series

$$u_{n_1} + u_{n_2} + u_{n_3} + \ldots,$$

in such a way as to make the sum

$$u_{p_1} + \ldots + u_{p_a} + u_{n_1} + \ldots + u_{n_b} + u_{p_{a+1}} + \ldots + u_{p_{a+c}} + u_{n_{b+1}} + \ldots + u_{n_{b+d}}$$

less than S again; and so on.

Proceeding in this way, we obtain a series whose sum at any stage of

the process, differs from S by less than the last term included. But the terms of the series

$$u_1 + u_2 + u_3 + \dots$$

are ultimately indefinitely small, since the series is convergent; we can therefore in this way obtain a series

$$u_{p_1} + \dots + u_{p_2} + u_{n_1} + \dots$$

whose sum differs from S by as little as we please; and it consists of the terms of the original series, disposed in a different order. This establishes the result above stated.

Corollary. If the terms of the original series are complex, they can be disposed in such an order as to give an arbitrarily assigned value to either the real or the imaginary part of the sum.

18. *Cauchy's theorem on the multiplication of absolutely convergent series.*

We shall now shew that *if two series*

$$S = u_1 + u_2 + u_3 + \dots$$

and

$$T = v_1 + v_2 + v_3 + \dots$$

are absolutely convergent, then the series

$$P = u_1 v_1 + u_2 v_1 + u_1 v_2 + \dots,$$

formed by the products of their terms, written in any order, is absolutely convergent, and has for sum ST.

Suppose that in order to include all the terms of the product

$$(u_1 + u_2 + u_3 + \dots + u_n)(v_1 + v_2 + \dots + v_n)$$

it is necessary to take m terms of P; and let k be any number greater than m.

Then

$$P_k = (u_1 + u_2 + \dots + u_n)(v_1 + v_2 + \dots + v_n) + \text{terms } u_\alpha v_\beta \text{ in which either } \alpha \text{ or } \beta \text{ is greater than } n,$$

so

$$|P_k - ST| \leqslant |S_n T_n - ST| + \text{terms } |u_\alpha|\,|v_\beta|.$$

Let $(n + p)$ be the greatest suffix contained in these suffixes α and β. Then

$$|P_k - ST| \leqslant |S_n T_n - ST| + \{|u_{n+1}| + \dots + |u_{n+p}|\}\{|v_1| + \dots + |v_{n+p}|\} + \{|u_1| + \dots + |u_n|\}\{|v_{n+1}| + \dots + |v_{n+p}|\}.$$

Now when n tends to infinity,

$$|u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}| \text{ tends to zero,}$$

and

$$|v_{n+1}| + \dots + |v_{n+p}| \text{ tends to zero,}$$

while their coefficients tend to finite limits.

Therefore $|P_k - ST|$ tends to zero, which proves the theorem.

Example 1. Shew that the series obtained by multiplying the two series

$$1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\frac{z^4}{2^4}+\dots,$$

and

$$1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots,$$

converges so long as the representative point of z lies in the ring-shaped region bounded by the circles $|z|=1$ and $|z|=2$.

For the first series converges only when $|z|<2$, and the second only when $|z|>1$, and both must converge if the product is to converge.

Example 2. Prove by multiplication of series that

$$\left\{\cos z+\frac{\cos 3z}{3^2}+\frac{\cos 5z}{5^2}+\dots\right\}\left\{\frac{\pi^2}{9}-\frac{2}{3}\left(\frac{\cos 2z}{2^2}+\frac{\cos 4z}{4^2}+\dots\right)\right\}=\cos z+\frac{\cos 3z}{3^4}+\frac{\cos 5z}{5^4}+\dots.$$

For the coefficient of $\cos(2r+1)z$ in the product on the left-hand side of the equation is

$$\frac{\pi^2}{9(2r+1)^2}-\frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\left\{\frac{1}{(2k-2r-1)^2}+\frac{1}{(2k+2r+1)^2}\right\},$$

or

$$\frac{\pi^2}{9(2r+1)^2}-\frac{1}{3(2r+1)^2}\sum_{k=1}^{\infty}\left\{\left(\frac{1}{2k-2r-1}-\frac{1}{2k}\right)^2+\left(\frac{1}{2k}-\frac{1}{2k+2r+1}\right)^2\right\},$$

or

$$\frac{\pi^2}{9(2r+1)^2}-\frac{1}{3(2r+1)^2}\sum_{k=1}^{\infty}\left\{\frac{2}{(2k)^2}+\frac{2}{(2k-1)^2}-\frac{4}{(2k-2r-1)(2k+2r+1)}\right\}+\frac{1}{3(2r+1)^4},$$

or

$$\frac{\pi^2}{9(2r+1)^2}+\frac{1}{3(2r+1)^4}-\frac{2}{3(2r+1)^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)+\frac{2}{3(2r+1)^4},$$

or

$$\frac{\pi^2}{9(2r+1)^2}+\frac{1}{(2r+1)^4}-\frac{2}{3(2r+1)^2}\cdot\frac{\pi^2}{6},$$

or

$$\frac{1}{(2r+1)^4},$$

which gives the required result.

19. *Mertens' theorem on the multiplication of a semi-convergent series by an absolutely convergent series.*

We shall now shew that *if a series*

$$S=u_1+u_2+u_3+\dots$$

is semi-convergent, and another series

$$T=v_1+v_2+v_3+\dots$$

is absolutely convergent, then the series

$$P=p_1+p_2+p_3+\dots$$

where $p_n=u_1v_n+u_2v_{n-1}+\dots+u_nv_1,$

is convergent, and its sum is ST.

For $P_n=$ the sum of all terms $u_\alpha v_\beta$ in which $\alpha+\beta<n+1$

$$=(u_1+u_2+\dots+u_n)(v_1+v_2+\dots+v_n)-v_2u_n-v_3(u_n+u_{n-1})-\dots$$
$$-v_n(u_2+u_3+\dots+u_n).$$

Therefore

$$|P_n - ST| \leqslant |S_n T_n - ST| + |u_n| |v_2| + |v_3| |u_n + u_{n-1}| + \dots$$
$$+ |v_n| |u_2 + u_3 + \dots + u_n|.$$

Now let k denote some number about half-way between 1 and n; let ϵ be the greatest of the quantities

$$|u_n|, \ |u_n + u_{n-1}|, \ \dots |u_n + u_{n-1} + \dots + u_{n-k}|,$$

and let γ be the greatest of the quantities

$$|u_n + \dots + u_{n-k-1}|, \ \dots |u_n + u_{n-1} + \dots + u_2|.$$

Then

$$|P_n - ST| \leqslant |S_n T_n - ST| + \epsilon \{|v_2| + |v_3| + \dots + |v_{k+2}|\} + \gamma \{|v_{k+3}| + \dots + |v_n|\}.$$

As n tends to infinity, ϵ and $\{|v_{k+3}| + \dots + |v_n|\}$ are infinitesimal, while $\{|v_2| + \dots + |v_{k+2}|\}$ and γ are finite. So every term on the right-hand side of the last equation is infinitesimal, and therefore in the limit

$$P = ST,$$

which establishes the theorem.

20. *Abel's result on the multiplication of series.*

We shall next prove a still more general theorem due to Abel*, which may be stated thus:

Let two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge to the limits U and V respectively, and let the quantity

$$u_1 v_n + u_2 v_{n-1} + \dots + u_n v_1$$

be denoted by w_n. Then *if the series*

$$w_1 + w_2 + w_3 + w_4 + \dots$$

converges at all, it converges to the sum UV.

It will be noticed that none of the series considered need be absolutely convergent.

We shall follow a method of proof due to Cesaro†.

Lemma I. If a set of quantities $s_1, s_2, s_3, \dots$ *tend to a limit* s, *then*

$$\operatorname*{Limit}_{n=\infty} \frac{1}{n} \sum_{i=1}^{n} s_i = s.$$

For if ϵ be any small positive number, we can find a number k such that the inequality

$$|s_r - s| < \epsilon$$

is satisfied for all values of r greater than k. We have therefore

$$\frac{1}{n} \sum_{i=1}^{n} s_i = \frac{1}{n} \sum_{i=1}^{k} s_i + \frac{1}{n} \sum_{i=k}^{i=n} s + \frac{1}{n} \sum_{i=k}^{n} (s_i - s).$$

* *Crelle's Journal*, I. (1827).

† *Bulletin des Sciences math.* (2) XIV. (1890).

Thus $$\left|\frac{1}{n}\sum_{i=1}^{n} s_i - \frac{n-k+1}{n} s\right| \leqslant \frac{1}{n}\sum_{i=1}^{k} |s_i| + \frac{1}{n}\sum_{i=k}^{n} |s_i - s|$$
$$< \frac{1}{n}\sum_{i=1}^{k} |s_i| + \frac{n-k+1}{n}\epsilon.$$

Now make n infinitely great compared with k; then $\frac{1}{n}\sum_{i=1}^{k} |s_i|$ tends to zero, and $\frac{n-k+1}{n}$ tends to unity,

and so $$\left|\operatorname*{Limit}_{n=\infty} \frac{1}{n}\sum_{i=1}^{n} s_i - s\right| < \epsilon;$$

and as ϵ can be made as small as we please, this establishes the Lemma.

Lemma II. If, as n increases indefinitely, a_n and b_n tend respectively to the limits a and b, then

$$\operatorname*{Limit}_{n=\infty} \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) = ab.$$

To prove this, let ν be the greatest integer contained in $\frac{1}{2}n$. Then if ϵ be any small positive number, we can take n so great that the inequality

$$|b_r - b| < \epsilon$$

holds so long as $$r > n - \nu.$$

Hence $$|a_1(b_n - b) + a_2(b_{n-1} - b) + \ldots + a_\nu(b_{n-\nu+1} - b)|$$
$$< \epsilon\{|a_1| + |a_2| + \ldots + |a_\nu|\}.$$

Hence $$\operatorname*{Limit}_{n=\infty} \frac{1}{n}|a_1(b_n - b) + a_2(b_{n-1} - b) + \ldots + a_\nu(b_{n-\nu+1} - b)|$$
$$< \epsilon \operatorname*{Limit}_{n=\infty} \frac{1}{n}\{|a_1| + |a_2| + \ldots + |a_\nu|\}$$
$$< \epsilon|a|, \text{ by Lemma I.}$$

The right-hand side of this inequality can be made as small as we please; hence

$$\operatorname*{Limit}_{n=\infty} \frac{1}{n}\{a_1(b_n - b) + a_2(b_{n-1} - b) + \ldots + a_\nu(b_{n-\nu+1} - b)\} = 0,$$

or $$\operatorname*{Limit}_{n=\infty} \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \ldots + a_\nu b_{n-\nu+1})$$
$$= \tfrac{1}{2}b \times \operatorname*{Limit}_{\nu=\infty} \frac{1}{\nu}(a_1 + a_2 + \ldots + a_\nu)$$
$$= \tfrac{1}{2}ab, \text{ by Lemma I.}$$

Similarly

$$\operatorname*{Limit}_{n=\infty} \frac{1}{n}(a_{\nu+1} b_{n-\nu} + a_{\nu+2} b_{n-\nu+1} + \ldots + a_n b_1) = \tfrac{1}{2}ab.$$

Adding the last two equations, we have

$$\underset{n=\infty}{\text{Limit}}\,(a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1) = ab,$$

which establishes Lemma II.

Now let W_n denote the sum of the n first terms of the series

$$w_1 + w_2 + w_3 + \dots,$$

considered in the above enunciation of Abel's result, we have

$$W_n = u_1 V_n + u_2 V_{n-1} + \dots + u_n V_1,$$

where U_n and V_n are used to denote the sums of the first n terms of the series U and V. From this we have

$$W_1 + W_2 + \dots + W_n = U_1 V_n + U_2 V_{n-1} + \dots + U_n V_1,$$

and so by Lemma II. it follows that

$$\underset{n=\infty}{\text{Limit}}\,\frac{1}{n}(W_1 + W_2 + \dots + W_n) = UV.$$

But if the set of quantities W_1, W_2, W_3, ... tend to a limit W, we have by Lemma I.

$$\underset{n=\infty}{\text{Limit}}\,\frac{1}{n}(W_1 + W_2 + \dots + W_n) = W.$$

Hence $$W = UV,$$

which establishes Abel's result.

Example 1. Shew that the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

is convergent, but that its square (formed by Abel's rule),

$$1 - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \dots$$

is divergent.

Example 2. If the convergent series

$$S = 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \dots$$

be multiplied by itself, the terms of the product being arranged as in Abel's result, shew that the resulting series is divergent if $r \leqslant \frac{1}{2}$, but that it converges to the sum S^2 when $r < \frac{1}{2}$.

(Cauchy and Cajori.)

21. *Power-Series.*

A series of the type

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots,$$

in which the quantities a_0, a_1, a_2, a_3 ... are independent of z, is called a *series proceeding according to ascending powers of* z, or briefly a *power-series.*

We shall now shew that *if a power-series converges for any value z_0 of z, it will be absolutely convergent for all values of z whose representative points are within a circle, which passes through z_0 and has its centre at the origin.*

For if z be such a point, we have $|z| < |z_0|$. Now since $\sum_{n=0}^{\infty} a_n z_0^n$ converges, the quantity $a_0 z_0^n$ must tend to zero as n increases indefinitely, and so we can write

$$|a_n| = \frac{\epsilon_n}{|z_0|^n},$$

where ϵ_n tends to zero as n increases. Thus

$$|a_0| + |a_1||z| + |a_2||z|^2 + \ldots = \epsilon_0 + \epsilon_1 \left|\frac{z}{z_0}\right| + \epsilon_2 \left|\frac{z}{z_0}\right|^2 + \epsilon_3 \left|\frac{z}{z_0}\right|^3 + \ldots.$$

Now ultimately every term in the series on the right-hand side is less than the corresponding term in the convergent geometric series

$$\sum_{n=0}^{\infty} \left|\frac{z}{z_0}\right|^n;$$

the series is therefore convergent; and so the power-series is *absolutely* convergent, as the series of moduli of its terms is a convergent series; which establishes the result stated.

It follows from this that the area in the z-plane over which a power-series converges must always be a *circle*; for if the series converges for any point outside the particular circle which has just been found, we can (by taking this point as the point z_0) obtain a new and larger circle within which the series will converge.

The circle in the z-plane which includes all the values of z for which the power-series

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

converges, is called the *circle of convergence* of the series. The radius of the circle is called the *radius of convergence*.

The radius of convergence of a power-series may be infinitely great; as happens for instance in the case of the series

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,$$

which represents the function $\sin z$; in this case the series converges for all finite values of z real or complex, i.e. over the whole z-plane.

On the other hand, the radius of convergence of a power-series may be infinitely small; thus in the case of the series

$$1 + 1!\,z + 2!\,z^2 + 3!\,z^3 + 4!\,z^4 + \ldots,$$

we have
$$\left|\frac{u_{n+1}}{u_n}\right| = n|z|,$$

which, for all values of n after some fixed value, is greater than unity when z has any value different from zero. The series converges therefore only at the point $z=0$, and its circle of convergence is infinitely small.

A power-series may or may not converge for points which are actually *on* the circumference of the circle; thus the series

$$1+\frac{z}{1^s}+\frac{z^2}{2^s}+\frac{z^3}{3^s}+\frac{z^4}{4^s}+\dots,$$

whose radius of convergence is unity, converges or diverges at the point $z=1$ according as s is greater or not greater than unity, as was seen in § 9.

22. *Convergency of series derived from a power-series.*

Let $$a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+\dots$$
be a power-series, and consider the series

$$a_1+2a_2z+3a_3z^2+4a_4z^3+\dots,$$

which is obtained by differentiating the power-series term by term. We shall now shew that *the derived series has the same circle of convergence as the original series.*

For let z be a point within the circle of convergence of the power-series; and choose a positive quantity r, intermediate in value between $|z|$ and the radius of convergence. Then, since the series $\sum_{n=0}^{\infty} a_n r^n$ converges absolutely, its terms must decrease indefinitely as n increases; and it must therefore be possible to find a positive quantity M, independent of n, such that the inequality

$$|a_n|<\frac{M}{r^n}$$

is true for all values of n.

Then the terms of the series

$$\sum_{n=1}^{\infty} n\,|a_n|\,|z|^{n-1}$$

are less than the corresponding terms of the series

$$\frac{M}{r}\sum_{n=1}^{\infty}\frac{n\,|z|^{n-1}}{r^{n-1}}.$$

But in this series we have

$$\frac{u_{n+1}}{u_n}=\frac{n+1}{n}\frac{|z|}{r}=\left(1+\frac{1}{n}\right)\frac{|z|}{r},$$

which, for all values of n greater than some fixed value, is constantly less than unity; this comparison-series therefore converges, and so the series

$$\sum_{n=1}^{\infty} n\,|a_n|\,|z|^{n-1}$$

converges; that is, the series $\sum_{n=1}^{\infty} na_n z^{n-1}$ converges absolutely for all points z situated within the circle of convergence of the original series $\sum_{n=0}^{\infty} a_n z^n$, and the two series have the same circle of convergence.

Similarly it can be shewn that the series $\sum_{n=0}^{\infty} \frac{a_n z^{n+1}}{n+1}$, which is obtained by integrating the original power-series term by term, has the same circle of convergence as $\sum_{n=0}^{\infty} a_n z^n$.

23. *Infinite Products.*

We proceed now to the consideration of another class of analytical expressions, known as *infinite products.*

Let $1 + a_1$, $1 + a_2$, $1 + a_3$, ... be an infinite set of quantities. If as n increases indefinitely, the product

$$(1 + a_1)(1 + a_2)(1 + a_3) \dots (1 + a_n)$$

(which we may denote by Π_n) tends to a definite limit other than zero, this is called the value of the infinite product

$$\Pi = (1 + a_1)(1 + a_2)(1 + a_3) \dots,$$

and the product is said to be *convergent.*

The product is often written $\prod_{n=1}^{\infty} (1 + a_n)$.

If the value of the product is independent of the order in which the factors occur, the convergence of the product is said to be *absolute.*

The condition for absolute convergence is given by the following theorem: *in order that the infinite product*

$$(1 + a_1)(1 + a_2)(1 + a_3) \dots$$

may be absolutely convergent, it is necessary and sufficient that the series

$$a_1 + a_2 + a_3 + \dots$$

should be absolutely convergent.

For $$\Pi_n = e^{\log(1+a_1)+\log(1+a_2)+\dots+\log(1+a_n)},$$

so that Π is absolutely convergent or not according as the series

$$\log(1 + a_1) + \log(1 + a_2) + \log(1 + a_3) + \dots$$

is absolutely convergent or not. But since $\log(1 + a_r)$ is nearly equal to a_r when a_r is small, the terms of this series always bear finite ratios to the corresponding terms of the series

$$a_1 + a_2 + a_3 + \dots,$$

and so the absolute convergence of one series entails that of the other; which establishes the result*.

* A discussion of the convergence of infinite products, in which the results are derived without making use of the logarithmic function, is given by Pringsheim, *Math. Ann.* XXXIII. pp. 119—154.

Example. Shew that the infinite product

$$\frac{\sin z}{z} \cdot \frac{\sin \frac{1}{2}z}{\frac{1}{2}z} \cdot \frac{\sin \frac{1}{3}z}{\frac{1}{3}z} \cdot \frac{\sin \frac{1}{4}z}{\frac{1}{4}z} \cdots$$

is absolutely convergent for all values of z.

For when n is large, $\dfrac{\sin \frac{z}{n}}{\frac{z}{n}}$ is of the form $1 - \dfrac{\lambda_n}{n^2}$, where λ_n is finite; and the series $\sum\limits_{n=1}^{\infty} \dfrac{\lambda_n}{n^2}$ is absolutely convergent, as is seen on comparing it with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$. The infinite product is therefore absolutely convergent.

24. *Some examples of infinite products.*

Consider the infinite product

$$\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\left(1 - \frac{z^2}{3^2\pi^2}\right)\cdots,$$

which represents the function $\dfrac{\sin z}{z}$.

In order to find whether it is absolutely convergent, we must consider the series $\sum\limits_{n=1}^{\infty} \dfrac{z^2}{n^2\pi^2}$, or $\dfrac{z^2}{\pi^2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$; this series is absolutely convergent, and so the product is absolutely convergent for all finite values of z.

But now let this product be written in the form

$$\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\cdots.$$

The absolute convergence of this product depends on that of the series

$$-\frac{z}{\pi} + \frac{z}{\pi} - \frac{z}{2\pi} + \frac{z}{2\pi} \cdots.$$

But this series is only semi-convergent, since its series of moduli

$$\frac{|z|}{\pi} + \frac{|z|}{\pi} + \frac{|z|}{2\pi} + \frac{|z|}{2\pi} + \cdots$$

is divergent. In this form therefore the infinite product is not absolutely convergent, i.e. if the order of the factors $\left(1 \pm \dfrac{z}{n\pi}\right)$ is deranged there is a risk of altering the value of the product.

Lastly, let the same product be written in the form

$$\left\{\left(1 - \frac{z}{\pi}\right) e^{\frac{z}{\pi}}\right\}\left\{\left(1 + \frac{z}{\pi}\right) e^{-\frac{z}{\pi}}\right\}\left\{\left(1 - \frac{z}{2\pi}\right) e^{\frac{z}{2\pi}}\right\}\left\{\left(1 + \frac{z}{2\pi}\right) e^{-\frac{z}{2\pi}}\right\}\cdots,$$

in which each of the expressions

$$\left(1 \pm \frac{z}{m\pi}\right) e^{\mp \frac{z}{m\pi}}$$

is counted as a single term of the infinite product. The absolute convergence of this product depends on that of the series

$$\left\{\log\left(1-\frac{z}{\pi}\right)+\frac{z}{\pi}\right\}+\left\{\log\left(1+\frac{z}{\pi}\right)-\frac{z}{\pi}\right\}+\left\{\log\left(1-\frac{z}{2\pi}\right)+\frac{z}{2\pi}\right\}+\dots,$$

or $$\left(-\frac{z^2}{2\pi^2}+\dots\right)+\left(-\frac{z^2}{2\pi^2}+\dots\right)+\left(-\frac{z^2}{2\pi^2 . 2^2}+\dots\right)+\left(-\frac{z^2}{2\pi^2 . 2^2}+\dots\right),$$

and the absolute convergence of this series follows from that of the series

$$1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{4^2}+\dots.$$

The infinite product in this last form is therefore again absolutely convergent, the adjunction of the factors $e^{\pm\frac{z}{n\pi}}$ having changed the convergence from conditional to absolute.

Example 1. Prove that $\prod_{n=1}^{\infty}\left\{\left(1-\frac{z}{c+n}\right)e^{\frac{z}{n}}\right\}$ is absolutely convergent for all values of z, if c is a constant other than a negative integer.

For the infinite product is absolutely convergent provided the series

$$\sum_{n=1}^{\infty}\left\{\left(1-\frac{z}{c+n}\right)e^{\frac{z}{n}}-1\right\} \text{ is,}$$

i.e. if $$\sum_{n=1}^{\infty}\left\{\left(1-\frac{z}{c+n}\right)\left(1+\frac{z}{n}+\frac{z^2}{2n^2}+\dots\right)-1\right\} \text{ is,}$$

i.e. if $$\sum_{n=1}^{\infty}\left\{\frac{zc-\frac{1}{2}z^2}{n^2}+\text{terms in } \frac{1}{n^3},\ \frac{1}{n^4} \text{ etc.}\right\} \text{ is,}$$

and on comparison with the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, this is seen to be the case.

Example 2. Shew that $\prod_{n=2}^{\infty}\left\{1-\left(1-\frac{1}{n}\right)^{-n}z^{-n}\right\}$ converges for all points z situated outside a circle whose centre is the origin and radius unity.

For the infinite product is absolutely convergent provided the series

$$\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)^{-n}z^{-n}$$

is absolutely convergent. But as n increases, $\left(1-\frac{1}{n}\right)^{-n}$ tends to the finite limit e, so the ratio of the $(n+1)$th term of the series to the nth term is ultimately $\frac{1}{z}$; there is therefore absolute convergence when $\left|\frac{1}{z}\right|<1$, or $|z|>1$.

Example 3. Shew that

$$\frac{1 . 2 . 3 \dots (n-1)}{z(z+1)(z+2)\dots(z+n-1)}n^z$$

tends to a finite limit as n increases indefinitely, unless z is a negative integer.

For the expression can be regarded as a product of which the nth term is

$$\frac{n}{z+n}\left(\frac{n+1}{n}\right)^z, \text{ or } \left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}, \text{ or } \left\{1+\frac{z(z-1)}{2n^2}+\text{terms in } \frac{1}{n^3}\right\}.$$

This product is therefore absolutely convergent, provided the series

$$\sum_{n=1}^{\infty}\left\{\frac{z(z-1)}{2n^2}+\text{terms in } \frac{1}{n^3}, \frac{1}{n^4}, \text{ etc.}\right\}$$

is absolutely convergent; and a comparison with the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ shews that this is the case. When z is a negative integer the expression clearly becomes infinite owing to the vanishing of one of the factors in the denominator.

Example 4. Prove that

$$z\left(1-\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{\pi}\right)\left(1-\frac{z}{3\pi}\right)\left(1-\frac{z}{4\pi}\right)\left(1+\frac{z}{2\pi}\right)\ldots=e^{-\frac{z}{\pi}\log 2}\sin z.$$

For the given product

$$=\underset{k=\infty}{\text{Limit}}\; z\left(1-\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{\pi}\right)\ldots\left(1-\frac{z}{(2k-1)\pi}\right)\left(1-\frac{z}{2k\pi}\right)\left(1+\frac{z}{k\pi}\right)$$

$$=\underset{k=\infty}{\text{Limit}}\left[\begin{array}{l} e^{\frac{z}{\pi}\left(-1-\frac{1}{2}+1-\frac{1}{3}-\frac{1}{4}+\frac{1}{2}-\ldots-\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{k}\right)} \\ \times z\left(1-\frac{z}{\pi}\right)e^{\frac{z}{\pi}}.\left(1-\frac{z}{2\pi}\right)e^{\frac{z}{2\pi}}\ldots\left(1-\frac{z}{2k\pi}\right)e^{\frac{z}{2k\pi}}.\left(1+\frac{z}{k\pi}\right)e^{-\frac{z}{k\pi}}\end{array}\right]$$

$$=\underset{k=\infty}{\text{Limit}}\; e^{-\frac{z}{\pi}\left(1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2k-1}-\frac{1}{2k}\right)}\, z\left(1-\frac{z}{\pi}\right)e^{\frac{z}{\pi}}\left(1+\frac{z}{\pi}\right)e^{-\frac{z}{\pi}}\left(1-\frac{z}{2\pi}\right)e^{\frac{z}{2\pi}}\left(1+\frac{z}{2\pi}\right)e^{-\frac{z}{2\pi}}\ldots,$$

since the product whose factors are

$$\left(1-\frac{z}{r\pi}\right)e^{\frac{z}{r\pi}}$$

is *absolutely* convergent and so the order of its factors can be altered.

Since $$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\ldots,$$

this shews that the given product is equal to

$$e^{-\frac{z}{\pi}\log 2}\sin z.$$

25. *Cauchy's theorem on products which are not absolutely convergent.*

We shall now shew that *if*

$$a_1+a_2+a_3+a_4+\ldots$$

is a semi-convergent series of real terms, then the infinite product

$$(1+a_1)(1+a_2)(1+a_3)\ldots$$

converges (though not absolutely) or diverges (to the value zero), according as the series

$$a_1^2 + a_2^2 + a_3^2 + \ldots$$

is convergent or divergent.

For the infinite product in question converges (though not absolutely) or diverges (to the value zero) according as the series

$$\log(1 + a_1) + \log(1 + a_2) + \ldots$$

is semi-convergent or diverges to the value $-\infty$.

Now since the series $\sum\limits_{n=1}^{n=\infty} a_n$ is convergent, the quantities a_n ultimately diminish indefinitely, and therefore we can write

$$\log(1 + a_n) = a_n - \frac{a_n^2}{2}(1 + \epsilon_n),$$

where $|\epsilon_n|$ tends to zero as n tends to infinity.

If the series $\sum\limits_{n=1}^{n=\infty} a_n^2$ diverges, it is clear therefore that the series $\Sigma \log(1 + a_n)$ must diverge to the value $-\infty$; if on the other hand the series $\sum\limits_{n=1}^{n=\infty} a_n^2$ converges, the series $\sum\limits_{n=1}^{n=\infty} \log(1 + a_n)$ is convergent. From this the results relating to the infinite product follow at once.

26. *Infinite Determinants.*

Infinite series and infinite products are not by any means the only known cases of infinite processes which can lead to convergent results. The researches of Mr G. W. Hill in the Lunar Theory* brought into notice the possibilities of *infinite determinants.*

The actual investigation of the convergence is due not to Hill but to Poincaré, *Bull. de la Soc. Math. de France*, XIV. (1886), p. 87. We shall follow the exposition given by H. von Koch, *Acta Math.* XVI. (1892), p. 217.

Let A_{ik} $(i, k = -\infty, \ldots +\infty)$ be a doubly-infinite set of given numbers, and denote by

$$D_m = [A_{ik}]_{i, k = -m, \ldots +m},$$

the determinant formed of the quantities A_{ik} $(i, k = -m - \ldots + m)$; then if, for indefinitely increasing values of m, the quantity D_m has a determinate limit D, we shall say that the infinite determinant

$$[A_{ik}]_{i, k = -\infty \ldots +\infty}$$

is *convergent* and has a value D. In the case in which the limit D does not exist, the determinant in question will be said to be *divergent*.

* Reprinted in *Acta Mathematica*, VIII. pp. 1—36 (1886).

The elements $A_{ii}\,(i = -\infty \ldots +\infty)$ are said to form the *principal diagonal* of the determinant D; the elements $A_{ik}\,(k = \infty \ldots +\infty)$ are said to form the *line* i; and the elements $A_{ik}\,(i = -\infty \ldots +\infty)$ are said to form the *column* k. Any element A_{ik} is called a *diagonal* or a *non-diagonal* element, according as $i = k$ or $i \lessgtr k$. The element $A_{0,0}$ is called the *origin* of the determinant.

27. *Convergence of an infinite determinant.*

We shall now shew that *an infinite determinant converges, provided the product of the diagonal elements converges absolutely and the sum of the non-diagonal elements converges absolutely.*

For let the diagonal elements of an infinite determinant D be denoted by $1 + a_{ii}\,(i = -\infty \ldots +\infty)$, and let the non-diagonal elements be denoted by $a_{ik}\left(i \lessgtr k,\ \begin{matrix} i = -\infty \ldots +\infty \\ k = -\infty \ldots +\infty \end{matrix}\right)$, so that the determinant is

$$\left|\begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 + a_{-1-1} & a_{-10} & a_{-11} & \cdots \\ \cdots & a_{0-1} & 1 + a_{00} & a_{01} & \cdots \\ \cdots & a_{1-1} & a_{10} & 1 + a_{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}\right|.$$

Then since the series

$$\sum_{i,k=-\infty}^{\infty} |a_{ik}|$$

is convergent, the product

$$\bar{P} = \prod_{i=-\infty}^{\infty} \left(1 + \sum_{k=-\infty}^{\infty} |a_{ik}|\right)$$

is convergent.

Now form the products

$$P_m = \prod_{i=-m}^{m} \left(1 + \sum_{k=-m}^{m} a_{ik}\right),$$

$$\bar{P}_m = \prod_{i=-m}^{m} \left(1 + \sum_{k=-m}^{m} |a_{ik}|\right);$$

then if, in the expansion of P_m, certain terms are replaced by zero and certain other terms have their signs changed, we shall obtain D_m; thus, to each term in the expansion of D_m there corresponds in the expansion of $\bar{P}_m$ a term of equal or greater modulus. Now $D_{m+p} - D_m$ represents the sum of those terms in the determinant D_{m+p} which vanish when the quantities $a_{ik}\,\{i, k = \pm(m+1) \ldots \pm(m+p)\}$ are replaced by zero; and to each of these terms there corresponds a term of equal or greater modulus in $\bar{P}_{m+p} - \bar{P}_m$.

Hence
$$|D_{m+p} - D_m| \leqslant \bar{P}_{m+p} - \bar{P}_m.$$

As the quantities $\bar{P}_m, \bar{P}_{m+1}, \ldots$ tend to a fixed limit, the quantities $D_m, D_{m+1}, \ldots$ will therefore tend to a fixed limit. This establishes the proposition.

28. We shall now shew that *a determinant, of the convergent form already considered, remains convergent when the elements of any line are replaced by any set of quantities whose moduli are all less than some fixed positive number.*

Replace, for example, the elements

$$\ldots A_{0,-m}, \ldots \quad A_0 \ldots A_{0,m} \ldots$$

of the line 0 by the quantities

$$\ldots \mu_{-m}, \ldots \quad \mu_0 \ldots \mu_m \ldots$$

which satisfy the inequality

$$|\mu_r| < \mu,$$

where μ is a positive number; and let the new values of D_m and D be denoted by D_m' and D'. Moreover, denote by $\bar{P}_m'$ and P' the products obtained in suppressing in P_m and $\bar{P}$ the factor corresponding to the index zero; we see that no term of D_m' can have a greater modulus than the corresponding term in the expansion of $\mu P_m'$; and consequently, reasoning as in the last article, we have

$$|D'_{m+p} - D_m'| \leqslant \mu \bar{P}'_{m+p} - \mu P_m',$$

which establishes the result stated.

Example. Shew that the necessary and sufficient condition for the absolute convergence of the infinite determinant

$$\begin{vmatrix} 1 & a_1 & 0 & 0 & \ldots \\ \beta_1 & 1 & a_2 & 0 & \ldots \\ 0 & \beta_2 & 1 & a_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

is that the series

$$a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + \ldots$$

shall be absolutely convergent. (von Koch.)

Miscellaneous Examples.

1. Find the range of values of z for which the series

$$2\sin^2 z - 4\sin^4 z + 8\sin^6 z - \ldots + (-1)^{n+1} 2^n \sin^{2n} z + \ldots$$

is convergent.

2. Shew that the series

$$\frac{1}{z} - \frac{1}{z+1} + \frac{1}{z+2} - \frac{1}{z+3} + \ldots$$

is semi-convergent, except for certain exceptional values of z; but that the series

$$\frac{1}{z} + \frac{1}{z+1} + \ldots + \frac{1}{z+p-1} - \frac{1}{z+p} - \frac{1}{z+p+1} - \ldots - \frac{1}{z+2p+q-1} + \frac{1}{z+2p+q} + \ldots,$$

in which $(p+q)$ negative terms always follow p positive terms, is divergent. (Simon.)

3. Shew that the series

$$\frac{1}{1^\alpha}+\frac{1}{2^\beta}+\frac{1}{3^\alpha}+\frac{1}{4^\beta}+\dots \qquad (1<\alpha<\beta)$$

is convergent. (Cesaro.)

4. Shew that the series

$$\alpha+\beta^2+\alpha^3+\beta^4+\dots \qquad (0<\alpha<\beta<1)$$

is convergent. (Cesaro.)

5. Shew that the series

$$\sum_{n=1}^{\infty} \frac{nz^{n-1}\left\{\left(1+\frac{1}{n}\right)^n-1\right\}}{(z^n-1)\left\{z^n-\left(1+\frac{1}{n}\right)^n\right\}}$$

converges absolutely for all values of z, except the values

$$z=\left(1+\frac{a}{m}\right)e^{\frac{2ki\pi}{m}}$$

$$(a=0,\ 1;\ k=0,\ 1,\ \dots m-1;\ m=1,\ 2,\ \dots \infty).$$

6. If s_n denote the sum of the first n terms of a convergent series whose sum is s, shew that

$$\underset{a=\infty}{\text{Limit}}\ e^{-a}\left\{s_0+s_1\frac{a}{1}+s_2\frac{a^2}{2!}+s_3\frac{a^3}{3!}+\dots\right\}=s.$$

7. In the series whose general term is

$$u_n=q^{n-\nu}x^{\frac{\nu(\nu+1)}{2}}, \qquad (0<q<1<x)$$

where ν denotes the number of figures in the expression of n in the ordinary decimal scale of notation, shew that

$$\underset{n=\infty}{\text{Limit}}\ u_n^{\frac{1}{n}}=q,$$

and that the series is convergent, although the quantity $\frac{u_{n+1}}{u_n}$ is infinitely great when n is infinitely great and of the form $1+10^{\nu-1}$. (Lerch.)

8. Shew that the series

$$q_1+q_1^2+q_2^3+q_1^4+q_2^5+q_3^6+q_1^7+\dots,$$

where $$q_n=q^{1+\frac{4}{n}}, \qquad (0<q<1)$$

is convergent, although the ratio of the $(n+1)$th term to the nth is greater than unity when n is not a triangular number. (Cesaro.)

9. Shew that the series

$$\sum_{n=0}^{\infty}\frac{e^{2n\pi ix}}{(w+n)^s},$$

where w is real, and where $(w+n)^s$ is understood to mean $e^{s\log(w+n)}$, the logarithm being taken in its arithmetic sense, is convergent for all values of s, when the imaginary part of x is positive, and is convergent for values of s whose real part is positive, when x is real.

10. Shew that the qth power of the convergent series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^r}$ is convergent when $\frac{q-1}{q}<r$, and divergent when $\frac{q-1}{q}>r$. (Cajori.)

11. If the two semi-convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^r} \text{ and } \sum \frac{(-1)^{n+1}}{n^s},$$

where r and s lie between 0 and 1, be multiplied together, and the product arranged as in Abel's result, shew that the necessary and sufficient condition for the convergence of the resulting series is $r+s>1$. (Cajori.)

12. Shew that if the series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots$$

be multiplied by itself any number of times, the terms of the product being arranged as in Abel's result, the resulting series converges. (Cajori.)

13. Shew that the qth power of the series

$$a_1 \sin\theta + a_2 \sin 2\theta + \ldots + a_n \sin n\theta + \ldots$$

is convergent whenever $\dfrac{q-1}{q}<r$, r being the maximum number satisfying the relation

$$a_n \leqslant \frac{1}{n^r}$$

for all values of n.

14. Shew that if θ is not equal to 0 or a multiple of 2π, and if the quantities $u_0, u_1, u_2, \ldots$ are all of the same sign and continually diminish in such a way that the limit of u_n is zero when n is infinite, then the series $\Sigma u_n \cos(n\theta+a)$ is convergent.

Shew also that, if the limit of u_n is not zero, but all the other conditions above are satisfied, the sum of the series is oscillatory if $\dfrac{\theta}{\pi}$ is commensurable, but that, if $\dfrac{\theta}{\pi}$ is incommensurable, the sum may have any value between certain limits whose difference is $a \operatorname{cosec} \tfrac{1}{2}\theta$, where a is the limit of u_n, when n is infinite.

(Cambridge Mathematical Tripos, 1896, Part I.)

15. Prove that

$$\prod_{n=1}^{\infty} \left\{ \left(1-\frac{z}{n}\right)^{n^k} e^{n^{k-1}z + n^{k-2}\frac{z^2}{2} + \ldots + \frac{z^{k+1}}{n(k+1)}} \right\},$$

where k is any positive integer, converges absolutely for all finite complex values of z.

16. Let $\sum\limits_{n=1}^{\infty} \theta_n$ be an absolutely convergent series. Shew that the infinite determinant

$$\Delta(c) = \left| \begin{array}{ccccccc} & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \ldots & \dfrac{(c-4)^2-\theta_0}{4^2-\theta_0} & \dfrac{-\theta_1}{4^2-\theta_0} & \dfrac{-\theta_2}{4^2-\theta_0} & \dfrac{-\theta_3}{4^2-\theta_0} & \dfrac{-\theta_4}{4^2-\theta_0} & \ldots \\ \ldots & \dfrac{-\theta_1}{2^2-\theta_0} & \dfrac{(c-2)^2-\theta_0}{2^2-\theta_0} & \dfrac{-\theta_1}{2^2-\theta_0} & \dfrac{-\theta_2}{2^2-\theta_0} & \dfrac{-\theta_3}{2^2-\theta_0} & \ldots \\ \ldots & \dfrac{-\theta_2}{0^2-\theta_0} & \dfrac{-\theta_1}{0^2-\theta_0} & \dfrac{c^2-\theta_0}{0^2-\theta_0} & \dfrac{-\theta_1}{0^2-\theta_0} & \dfrac{-\theta_2}{0^2-\theta_0} & \ldots \\ \ldots & \dfrac{-\theta_3}{2^2-\theta_0} & \dfrac{-\theta_2}{2^2-\theta_0} & \dfrac{-\theta_1}{2^2-\theta_0} & \dfrac{(c+2)^2-\theta_0}{2^2-\theta_0} & \dfrac{-\theta_1}{2^2-\theta_0} & \ldots \\ \ldots & \dfrac{-\theta_4}{4^2-\theta_0} & \dfrac{-\theta_3}{4^2-\theta_0} & \dfrac{-\theta_2}{4^2-\theta_0} & \dfrac{-\theta_1}{4^2-\theta_0} & \dfrac{(c+4)^2-\theta_0}{4^2-\theta_0} & \ldots \\ & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \end{array} \right|$$

converges: and shew that the equation

$$\Delta(c)=0$$

is equivalent to the equation

$$\sin^2 \tfrac{1}{2}\pi c = \Delta(0) \sin^2 \tfrac{1}{2}\pi\theta_0^{\frac{1}{2}}. \qquad \text{(Hill.)}$$

CHAPTER III.

The Fundamental Properties of Analytic Functions; Taylor's, Laurent's, and Liouville's Theorems.

29. *The dependence of one complex number on another.*

The problems with which Analysis is mainly occupied relate to the *dependence* of one complex number on another. If z and ζ are two complex numbers, so connected that the value of one of them is determined by the value of the other, e.g. if ζ is the square of z, then the two numbers are said to *depend* on each other.

This dependence must not be confused with the most important case of it, which will be explained later under the title of *analytic functionality*.

If ζ is a real function of a real variable z, then the relation between ζ and z, which may be written

$$\zeta = f(z),$$

can be visualised by a curve in a plane, namely the locus of a point whose coordinates referred to rectangular axes in the plane are (z, ζ). No such simple and convenient geometrical figure can be found for the purpose of visualising an equation

$$\zeta = f(z),$$

considered as defining the dependence of one complex number $\zeta = \xi + i\eta$ on another complex number $z = x + iy$. A representation strictly analogous to the one already given for real variables would require four-dimensional space, since the number of quantities ξ, η, x, y, is now four.

One suggestion (made by Lie and Weierstrass) is to use a doubly-manifold system of lines in the quadruply-manifold totality of lines in three-dimensional space.

Another suggestion is to represent ξ and η separately by means of surfaces

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

A third suggestion, due to Heffter*, is to write

$$\zeta = re^{i\theta},$$

then draw the surface $r = r(x, y)$—which may be called the *modular-surface* of the function—and on it to express the values of θ by surface-markings. It might be possible to modify this suggestion in various ways by representing θ by curves drawn on the surface $r = r(x, y)$.

* *Zeitschrift für Math. u. Phys.* XLIV. (1899), p. 235.

30. *Continuity.*

Let $f(z)$ be a quantity which, for all values of z lying within given limits, depends on z.

Let z_1 be a point situated within these limits. Then $f(z)$ is said to be *continuous* at the point z_1, if, corresponding to any given positive quantity ϵ, however small, a finite positive quantity η can be found, such that the inequality

$$|f(z)-f(z_1)|<\epsilon$$

is satisfied so long as $|z-z_1|$ is less than η.

If $f(z)$ is continuous at $z=z_1$, and if its real and imaginary parts be denoted by u and v, then u and v depend continuously on z.

For if $f(z)=u+iv$, we have

$$|(u-u_1)+i(v-v_1)|<\epsilon,$$

and so
$$(u-u_1)^2+(v-v_1)^2<\epsilon^2,$$

which gives
$$(u-u_1)^2<\epsilon^2 \text{ and } (v-v_1)^2<\epsilon^2,$$

and so
$$|u-u_1|<\epsilon \quad\text{and}\quad |v-v_1|<\epsilon.$$

The popular idea of continuity, so far as it relates to a real variable ζ depending on another real variable z, is somewhat different to that just considered, and may perhaps best be expressed by the definition "The quantity ζ is said to depend continuously on z if, as z passes through the series of all values intermediate between any two adjacent values z_1 and z_2, ζ passes through the series of all values intermediate between the corresponding values ζ_1 and ζ_2."

The question thus arises, how far this popular definition is equivalent to the analytical definition given above.

Cauchy shewed that if a real variable ζ, depending on a real quantity z, satisfies the analytical definition, then it also satisfies what we have called the popular definition. But the converse of this is not true, as was shewn by Darboux. This fact may be illustrated by the following example*.

Let $E(x)$ denote the integer next less than x; and let

$$f(x)=x\left[1-E\left\{\frac{1}{1+E(x^2)}\right\}\right]+E\left\{\frac{1}{1+E(x^2)}\right\}\sin\frac{\pi}{2x+E\left(\dfrac{1}{1+x^2}\right)}.$$

At $x=0$, we have $f(x)=0$.

Between $x=-1$ and $x=+1$ (except at $x=0$), we have

$$f(x)=\sin\frac{\pi}{2x}.$$

From this it is easily seen that $f(x)$ depends continuously on x near $x=0$, in the sense of the popular definition, but is not continuous in the sense of the analytical definition.

* Due to Mansion, *Mathesis*, IX. (1899).

31. *Definite integrals.*

Let z_0 and Z be any two values of z; and let their representative points A and B in the z-plane be connected by an arc (straight or curved) AB; and let $z_1, z_2, z_3, \ldots z_n$ be a number of points taken on the line AB in any manner.

Let $f(z)$ be a quantity which, for variations of z along the arc AB, depends continuously on z.

Let z_0' be any point situated in the interval z_0z_1 of the curve: let z_1' be any point situated in the interval z_1z_2: and so on: and consider the sum

$$s = f(z_0')(z_1 - z_0) + f(z_1')(z_2 - z_1) + \ldots + f(z_n')(Z - z_n).$$

We shall shew that *if the number n increases indefinitely, in such a way that each of the quantities $|z_r - z_{r-1}|$ tends to zero, then this sum will tend to a fixed limit, independently of the way in which the points*

$$z_1, z_2, \ldots z_n, \quad z_0', z_1', \ldots z_n',$$

are chosen.

For let ϵ be a given small positive quantity. Since $f(z)$ is continuous, for each point $z = a$ of the arc AB we can find a quantity η_a such that

$$|f(z) - f(a)| < \epsilon,$$

so long as $$|z - a| < \eta_a.$$

Let η be the least value of η_a corresponding to points a on the arc AB. We shall suppose the subdivision of the arc has been carried so far that each quantity $|z_r - z_{r-1}|$ is less than η, and shall first find the effect of putting in further subdivisions.

Suppose then that the interval z_0z_1 is subdivided at points $z_{01}, z_{02}, \ldots z_{0r_0}$; that the interval z_1z_2 is subdivided at the points $z_{11}, z_{12}, \ldots z_{1r_1}$; and so on: so that the sum s becomes

$$\begin{aligned} s' = {} & f(z_0'')(z_{01} - z_0) + f(z_{01}')(z_{02} - z_{01}) + \ldots \\ & + f(z_1'')(z_{11} - z_1) + f(z_{11}')(z_{12} - z_{11}) + \ldots \\ & + \ldots, \end{aligned}$$

where z_0'' is any point in the interval z_0z_{01}, z_{01}' is any point in the interval $z_{01}z_{02}$, and so on.

Then

$$\begin{aligned} s' - s = {} & \{f(z_0'') - f(z_0')\}(z_{01} - z_0) + \{f(z_{01}') - f(z_0')\}(z_{02} - z_{01}) + \ldots \\ & + \{f(z_1'') - f(z_1')\}(z_{11} - z_1) + \{f(z_{11}') - f(z_1')\}(z_{12} - z_{11}) + \ldots \\ & + \ldots. \end{aligned}$$

Therefore

$$|s'-s| < \epsilon \{ |z_{01}-z_0| + |z_{02}-z_{01}| + \dots \}$$

$< \epsilon \times$ the length of the broken line connecting the points $z_0, z_{01}, z_{02}, \dots$

$< \epsilon l,$

where l is the length of the arc AB.

Now by making ϵ indefinitely small, we can make the right-hand side of this equation as small as we please; and therefore the sum s tends to a definite limit when the number of subdivisions is indefinitely increased, provided that at each change in the subdivisions the old points of division are retained.

The restriction contained in the last phrase has still to be removed. To do this, suppose that two different methods of division, in each of which the quantities $|z_r - z_{r-1}|$ are less than η, furnish sums s_1 and s_2. Now combine the two methods of division, so that every point of division in either of the original schemes becomes a point of division in the new scheme. Let the sum corresponding to this new method of division be s_{12}. Then since by the above

$$|s_1 - s_{12}| < \epsilon l \quad \text{and} \quad |s_2 - s_{12}| < \epsilon l,$$

we have

$$|s_1 - s_2| < 2\epsilon l,$$

which shews that s_1 and s_2 tend to the same limit. The theorem is thus established.

The limit thus shewn to exist is called the *definite integral* of $f(z)$, taken along the arc AB; it is denoted by

$$\int_{AB} f(z)\,dz\,;$$

in cases where there is no ambiguity as to path, it may be denoted by

$$\int_A^B f(z)\,dz.$$

As an example* of the evaluation of a definite integral directly from the definition, suppose it is required to find the definite integral of the continuously dependent quantity $(1-z^2)^{-\frac{1}{2}}$, taken along the straight line (part of the real axis) joining the origin $(z=0)$ to a point $z=Z$, where Z is real. Denote the definite integral by I. Then by definition,

$$I = \operatorname*{Limit}_{n=\infty} \sum_{r=0}^{n} \frac{z_{r+1}-z_r}{(1-z_r'^2)^{\frac{1}{2}}},$$

and the mode of choosing the points z_r and z_r' is arbitrary, within the limits already explained; we shall take

$$z_r = \sin r\delta,$$

$$z_r' = \sin (r+\tfrac{1}{2})\,\delta,$$

where

$$\delta = \frac{1}{n+1} \sin^{-1} Z.$$

* Netto, *Zeitschrift für Math.* XL. (1895).

Thus
$$I = \underset{n=\infty}{\text{Limit}} \sum_{r=0}^{n} \frac{\sin(r+1)\delta - \sin r\delta}{\cos(r+\frac{1}{2})\delta}$$
$$= \underset{n=\infty}{\text{Limit}} \sum_{r=0}^{n} 2\sin\frac{\delta}{2}$$
$$= \underset{n=\infty}{\text{Limit}}\, 2(n+1)\sin\frac{\delta}{2}$$
$$= \sin^{-1} Z \underset{\delta=0}{\text{Limit}} \frac{\sin\frac{\delta}{2}}{\frac{\delta}{2}}$$
$$= \sin^{-1} Z.$$

The value of the definite integral is therefore $\sin^{-1} Z$.

32. *Limit to the value of a definite integral.*

Let M be the greatest value of $|f(z)|$ at points on the arc of integration AB.

Then
$$|f(z_0')(z_1 - z_0) + f(z_1')(z_2 - z_1) + \dots + f(z_n')(Z - z_n)|$$
$$\leqslant |f(z_0')|\,|z_1 - z_0| + |f(z_1')|\,|z_2 - z_1| + \dots + |f(z_n')|\,|Z - z_n|$$
$$\leqslant M\{|z_1 - z_0| + |z_2 - z_1| + \dots + |Z - z_n|\}$$
$$\leqslant Ml,$$
where l is the length of the arc of integration AB.

We see therefore, on proceeding to the limit, that
$$\left|\int_{AB} f(z)\,dz\right|$$
cannot be greater than the quantity Ml.

33. *Property of the elementary functions.*

The reader will be already familiar with the word *function*, as used (in text-books on Algebra, Trigonometry, and the Differential Calculus) to denote analytical expressions depending on a variable z; such for example as
$$z^2,\quad e^z,\quad \log z,\quad \sin^{-1} z^{\frac{3}{2}}.$$
These quantities, formed by combinations of the elementary functions of analysis, have in common a remarkable property, which will now be investigated.

Take as an example the function e^z.

Write
$$e^z = f(z).$$
Then if z' be a point near the point z, we have
$$\frac{f(z') - f(z)}{z' - z} = \frac{e^{z'} - e^z}{z' - z} = e^z \cdot \frac{e^{(z'-z)} - 1}{z' - z}$$
$$= e^z\left\{1 + \frac{z - z'}{2!} + \frac{(z - z')^2}{3!} + \dots\right\};$$

and hence, if the point z' tends to coincide with z, the limiting value of the quotient

$$\frac{f(z')-f(z)}{z'-z}$$

is e^z.

This shews that *the limiting value of*

$$\frac{f(z')-f(z)}{z'-z}$$

is in this case independent of the direction of the short path by which the point z' moves towards coincidence with z, i.e. it is independent of the direction in which z' lies as viewed from z.

It will be found that this property is shared by all the well-known elementary functions; namely, that if $f(z)$ be one of these functions and h be any small complex quantity, the limiting value of

$$\frac{1}{h}\{f(z+h)-f(z)\}$$

is independent of the mode in which h tends to zero.

34. *Occasional failure of the property.*

For each of the elementary functions, however, there will be certain points z at which this property will cease to hold good. Thus it does not hold for the function $\frac{1}{z-a}$ at the point $z=a$, since the limiting value of

$$\frac{1}{h}\left\{\frac{1}{z-a-h}-\frac{1}{z-a}\right\}$$

is not finite when $z=a$. Similarly it does not hold for the functions $\log z$ and $z^{\frac{1}{2}}$ at the point $z=0$.

These exceptional points are called *singular points* or *singularities* of the function $f(z)$ under consideration; at other points the function is said to be *regular.*

35. *The analytical function.*

The property noted in § 33 will be taken as the basis of our definition of an *analytic function,* which may be stated as follows.

Let an area in the z-plane be given; and let u be a quantity which has a definite finite value corresponding to every point z in that area. Let z, $z+\delta z$ be values of the variable z at two neighbouring points, and u, $u+\delta u$ the corresponding values of u. Then if at every point z within the area $\frac{\delta u}{\delta z}$ tends to a finite limiting value when δz tends to zero, independently of

the way in which δz tends to zero, u is said to be an *analytic function* of z, *regular* within the area.

We shall generally use the word "function" alone to denote an analytic function, as the functions studied in this work will be almost exclusively analytic functions.

In the foregoing definition, the function u has been defined only within a certain area in the z-plane. As will be seen subsequently, however, the function u will generally exist for other values of z not excluded in this area: and (as in the case of the elementary functions already discussed) may have *singularities*, for which the fundamental property no longer holds, at certain points outside the limits of the area.

The definition of functionality must now be translated into analytical language.

If $f(z)$ be a function of z, regular in the neighbourhood of a particular value z, then, by the definition, the quantity

$$\frac{f(z') - f(z)}{z' - z}$$

tends to a definite limit, depending only on z, when z' tends to z. Let this limit be denoted by the symbol $f'(z)$.

Then (by the definition of a limit) for every positive quantity ϵ, however small, it is possible to find a quantity η, such that

$$\left| \frac{f(z') - f(z)}{z' - z} - f'(z) \right|$$

is less than ϵ, so long as $|z' - z|$ is less than η.

If therefore we write

$$f(z') = f(z) + (z' - z) f'(z) + \epsilon' (z' - z),$$

we see that $|\epsilon'|$ is less than ϵ, so long as $|z' - z|$ is less than η; that is, the function $f(z)$ must be such that the quantity ϵ', defined by the equation

$$f(z') = f(z) + (z' - z) f'(z) + \epsilon (z' - z),$$

tends to the limit zero as z' tends to z.

The necessity for a strict definition of the term "function" may be seen from the following consideration.

Let y denote the temperature at a certain place at time t. As t varies, y will vary, and y may loosely be called a "function" of t. But y cannot be expressed in terms of t by a Maclaurin's infinite series

$$y = (y)_{t=0} + t \cdot \left(\frac{dy}{dt}\right)_{t=0} + \frac{t^2}{2!} \left(\frac{d^2y}{dt^2}\right)_{t=0} + \dots;$$

for if it could, the knowledge of the temperature for a single day would enable us to determine the quantities

$$(y)_{t=0}, \quad \left(\frac{dy}{dt}\right)_{t=0}, \quad \left(\frac{d^2y}{dt^2}\right)_{t=0}, \text{ etc.},$$

and then from the Maclaurin's expansion it would be possible to predict the temperature for the future!

Maclaurin's series is in fact, as will appear subsequently, applicable only to *analytic* functions, in the sense in which analytic functions have been defined above.

36. *Cauchy's theorem on the integral of a function round a contour.*

A simple closed curve in the plane of the variable z is often called a *contour*: if A, B, C, D be points taken in order along the arc of the contour, and if $f(z)$ be a quantity depending on z and continuous at all points on the arc, then the integral

$$\int_{ABCDA} f(z)\,dz,$$

taken round the contour, starting from the point A and returning to A again, is called *the integral of the quantity $f(z)$ taken round the contour.* Clearly the value of the integral taken round the contour is unaltered if some point in the contour other than A is taken as the starting-point.

We shall now prove a result due to Cauchy, which may be stated as follows. *If $f(z)$ is an analytic function, regular at all points in the interior of a contour, then*

$$\int f(z)\,dz = 0,$$

where the integration is taken round the contour.

For let A, B, C, D be points in order on the contour. Join A to C by an arc AEC, which will divide the region contained within the contour into two distinct portions. Then the integral taken round the contour $ABCDA$ is equal to the sum of the integrals taken round the two contours $ABCEA$ and $AECDA$; for

$$\begin{aligned}\int_{ABCEA} f(z)\,dz + \int_{AECDA} f(z)\,dz \\ &= \int_{ABC} f(z)\,dz + \int_{CEA} f(z)\,dz + \int_{AEC} f(z)\,dz + \int_{CDA} f(z)\,dz \\ &= \int_{ABCDA} f(z)\,dz,\end{aligned}$$

since the integrals along CEA and AEC neutralise each other.

Now join any point E on the arc AEC to D by an arc EFD, and join E to B by an arc EGB; then in the same way we see that the integral round $ABCEA$ is equal to the sum of the integrals round $ABGEA$ and $EGBCE$, and the integral round $AECDA$ is equal to the sum of the integrals round $AEFDA$ and $DFECD$.

Thus the original contour-integral is equal to the sum of the integrals

round the four contours $ABGEA$, $EGBCE$, $AEFDA$, $DFECD$, into which it has been divided by drawing the cross-lines.

Proceeding in this way by drawing more cross-lines, we see that the original contour-integral can be decomposed into the sum of any number of integrals round smaller contours, which constitute a network filling up the original contour.

Now suppose that each of these small contours has linear dimensions of the same order of magnitude as a small quantity l. Let z_0 be a point within one of them. Then on this small contour we have

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)\epsilon,$$

where ϵ is infinitely small when l is infinitely small.

Thus $$\int f(z)\,dz = \int f(z_0)\,dz + \int (z - z_0) f(z_0)\,dz + \int (z - z_0)\epsilon\,dz,$$

where all the integrals are taken round the small contour.

Now $$\int f(z_0)\,dz = f(z_0)\int dz$$

$$= f(z_0) \times \text{the increase in value of } z \text{ after once describing the small contour}$$

$$= 0.$$

Similarly $$\int f(z_0)(z - z_0)\,dz = \frac{1}{2} f(z_0) \int d\,\{(z - z_0)^2\} = 0,$$

when the integral is taken round the small contour.

Thus, if η be the greatest value of $|\epsilon|$ for points on the small contour, we have

$$\left|\int f(z)\,dz\right| \leqslant \eta \int |z - z_0|\,|dz|,$$

where the integrals are taken round the small contour.

Now the right-hand side of this equation is clearly of the order ηl^2 of small quantities. The value of $\int f(z)\,dz$, taken round the small contour, is therefore a small quantity of order ηl^2.

Now the number of such small contours in a given area is of the order $\frac{1}{l^2}$. If η' be the maximum value of η for all the small contours in the area, we see therefore that the total sum of the integrals for all the small contours in the area is at most of the order $\eta' l^2 \times \frac{1}{l^2}$ or η'; and η' can be made indefinitely small by decreasing l.

It follows, therefore, that the sum of the integrals round all the small

contours is zero; that is, the integral round the original contour is zero, which establishes Cauchy's result.

Corollary 1. If there are two paths z_0AZ and z_0BZ from z_0 to Z, and if $f(z)$ is a regular function of z at all points in the area enclosed by these two paths, then $\int_{z_0}^{Z} f(z)\,dz$ has the same value whether the path of integration is z_0AZ or z_0BZ. This follows from the fact that z_0AZBz_0 is a simple contour, and so the integral taken round it (which is the difference of the integrals along z_0AZ and z_0BZ) is zero. Thus, if $f(z)$ be an analytic function of z, the value of $\int_{AB} f(z)\,dz$ is to a certain extent independent of the choice of the arc AB, and depends only on the terminal points A and B. It must be borne in mind that *this is only the case when $f(z)$ is an analytical function* in the sense of § 35.

Corollary 2. Suppose that two simple closed curves C_0 and C_1 are given, such that C_0 completely encloses C_1, as e.g. would be the case if C_0 and C_1 were concentric circles or confocal ellipses.

Suppose moreover that $f(z)$ is an analytic function, which is regular at all points in the ring-shaped space contained between C_0 and C_1. Then by drawing a network of intersecting lines in this ring-shaped space, we can shew exactly as in the theorem just proved that *the integral*

$$\int f(z)\,dz$$

is zero, where the integration is taken round the whole boundary of the ring-shaped space; this boundary consisting of two curves C_0 and C_1, the one described in a positive (counter-clockwise) direction and the other described in a negative (clockwise) direction.

Corollary 3. And in general if any connected region be given in the z-plane, bounded by any number of curves C_0, C_1, C_2, ..., and if $f(z)$ be any function of z which is regular everywhere in this region, *then*

$$\int f(z)\,dz$$

is zero, where the integral is taken round the whole boundary of the region; this boundary consisting of the curves C_0, C_1, ..., each described in such a sense that the region is kept either always on the right or always on the left of a person walking in the sense in question round the boundary.

An extension of Cauchy's theorem $\int f(z)\,dz=0$, to curves lying on a cone whose vertex is at the origin, has been made by Raout (*Nouv. Annales de Math.* (3) XVI. (1897),

pp. 365–7). Osgood (*Bull. Amer. Math. Soc.*, 1896) has shewn that the property $\int f(z)\,dz=0$ may be taken as the defining-property of an analytic function, the other properties being deducible from it.

Example. A ring-shaped region is bounded by the two circles $|z|=1$ and $|z|=2$ in the z-plane. Verify that the value of $\int \frac{dz}{z}$, where the integral is taken round the boundary of this region, is zero.

For the boundary consists of the circumference $|z|=1$, described in the clockwise direction, together with the circumference $|z|=2$, described in the counter-clockwise direction. Thus if for points on the first circumference we write $z=e^{i\theta}$, and for points on the second circumference we write $z=2e^{i\phi}$, then θ and ϕ are real, and the integral becomes

$$\int_0^{-2\pi} \frac{i\,.\,e^{i\theta}\,d\theta}{e^{i\theta}} + \int_0^{2\pi} \frac{i\,.\,2e^{i\phi}\,d\phi}{2e^{i\phi}},$$

or

$$-2\pi i+2\pi i, \text{ i.e. zero.}$$

37. *The value of a function at a point, expressed as an integral taken round a contour enclosing the point.*

Let C be a contour within which $f(z)$ is a regular function of z.

Then if a be any point within the contour, the expression

$$\frac{f(z)}{z-a}$$

represents a function of z, which is regular at all points within the contour C except the point $z=a$, where it has a singularity.

Now with the point $z=a$ as centre, describe a circle γ of very small radius. Then in the ring-shaped space between γ and C, the function

$$\frac{f(z)}{z-a}$$

is regular, and so by Corollary 2 of the preceding article we have

$$\int_C \frac{f(z)\,dz}{z-a} - \int_\gamma \frac{f(z)\,dz}{z-a} = 0,$$

where $\int_C$ and $\int_\gamma$ denote integrals taken in the positive or counter-clockwise sense round the curves C and γ respectively.

But (§ 35) $$\int_\gamma \frac{f(z)\,dz}{z-a} = \int_\gamma \frac{f(a)+(z-a)f'(a)+\epsilon\,(z-a)}{z-a}\,dz,$$

where ϵ is a quantity which tends to zero when the radius of the circle γ is indefinitely diminished. Thus

$$\int_C \frac{f(z)\,dz}{z-a} = f(a)\int_\gamma \frac{dz}{z-a} + f'(a)\int_\gamma dz + \int_\gamma \epsilon\,dz.$$

Now if at points on the circumference γ we write

$$z - a = re^{i\theta},$$

where r is the radius of the circle γ, we have

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i,$$

and $$\int_\gamma dz = \int_0^{2\pi} ire^{i\theta}\,d\theta = 0;$$

also $$\left|\int_\gamma \epsilon\,dz\right| \leqslant \eta \,.\, 2\pi r,$$

where η is the greatest value of $|\epsilon|$ for points z on γ; and therefore in the limit when r is made indefinitely small we have

$$\int_\gamma \epsilon\,dz = 0.$$

Thus $$\int_C \frac{f(z)\,dz}{z-a} = 2\pi i f(a),$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}.$$

This remarkable result expresses the value of a function $f(z)$ at any point a *within* a contour C, in terms of an integral which depends only on the value of $f(z)$ at points *on* the contour itself.

Corollary. If $f(z)$ is a regular function of z in a ring-shaped region bounded by two curves C and C', and a is a point in the region, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z-a}\,dz,$$

where C is the outer of the curves and the integrals are taken in the positive or counter-clockwise sense.

38. *The Higher Derivates.*

The quantity $f'(z)$, which represents the limiting value of

$$\frac{f(z+h) - f(z)}{h}$$

when h tends to zero, is called the *derivate* of $f(z)$. We shall now shew that *$f'(z)$ is itself an analytic function of z, and consequently itself possesses a derivate.*

For if C be a contour surrounding the point z, and situated entirely within the region in which $f(z)$ is regular, we have

$$\begin{aligned} f'(a) &= \operatorname*{Limit}_{h=0} \frac{f(a+h)-f(a)}{h} \\ &= \operatorname*{Limit}_{h=0} \frac{1}{2\pi i h}\left\{\int_C \frac{f(z)\,dz}{z-a-h} - \int_C \frac{f(z)\,dz}{z-a}\right\} \\ &= \operatorname*{Limit}_{h=0} \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)(z-a-h)} \\ &= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2} + \operatorname*{Limit}_{h=0} \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2(z-a-h)}. \end{aligned}$$

Now
$$\int_C \frac{f(z)\,dz}{(z-a)^2(z-a-h)}$$
is a finite quantity, since the integrand
$$\frac{f(z)}{(z-a)^2(z-a-h)}$$
is finite at all points of the contour C, and the path of integration is of finite length. Hence

$$\operatorname*{Limit}_{h=0} \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2(z-a-h)} = 0,$$

and consequently
$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2},$$
a formula which expresses the value of the derivate of a function at a point as an integral taken round a contour enclosing the point.

From this formula we have, if h be any small quantity,

$$\begin{aligned} \frac{f'(a+h)-f'(a)}{h} &= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{h}\left\{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right\} \\ &= \frac{1}{2\pi i}\int_C f(z)\,dz \,.\, \frac{2\left(z-a-\frac{h}{2}\right)}{(z-a-h)^2(z-a)^2} \\ &= \frac{2}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^3} + hA, \end{aligned}$$

where A is a quantity which is easily seen to remain finite as h tends to zero.

Therefore as h tends to zero, the expression
$$\frac{f'(a+h)-f'(a)}{h}$$
tends to a limiting value, namely
$$\frac{2}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^3}.$$

The quantity $f'(a)$ is therefore an analytical function of a; its derivate, which is represented by the expression just given, is denoted by $f''(a)$, and is called the *second derivate* of $f(a)$.

Similarly it can be shewn that $f''(a)$ is an analytical function of a, possessing a derivate equal to

$$\frac{2\,.\,3}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^4};$$

this is denoted by $f'''(a)$, and is called the *third derivate* of $f(a)$. And in general an nth derivate of $f(a)$ exists, expressible by the integral

$$\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}},$$

and having a derivate of the form

$$\frac{(n+1)!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+2}};$$

this can be proved by induction in the following way.

Let
$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Then

$$\begin{aligned}\frac{f^{(n)}(a+h)-f^{(n)}(a)}{h} &= \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{h}\left\{\frac{1}{(z-a-h)^{n+1}}-\frac{1}{(z-a)^{n+1}}\right\}\\ &= \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}h}\left\{\left(1-\frac{h}{z-a}\right)^{-n-1}-1\right\}\\ &= \frac{(n+1)!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+2}}\\ &\qquad + \text{terms which vanish when } h \text{ tends to zero.}\end{aligned}$$

Thus
$$f^{(n+1)}(a)=\frac{(n+1)!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+2}},$$

which establishes the required result.

A function which possesses a first derivate at all points of a region in the z-plane therefore possesses derivates of all orders.

Example 1. Verify the theorem

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

by use of Taylor's Theorem.

By Taylor's Theorem we have

$$\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}=\frac{n!}{2\pi i}\int_C \frac{f(a)+(z-a)f'(a)+\ldots+\dfrac{(z-a)^n}{n!}f^{(n)}(a)+\ldots}{(z-a)^{n+1}}\,dz.$$

But when k is an integer other than unity, $\int_C \frac{dz}{(z-a)^k}$ is zero, since $\frac{1}{(z-a)^{k-1}}$ resumes its original value after describing the contour. So the only surviving part of the right-hand side is $\frac{1}{2\pi i} . f^{(n)}(a) \int_C \frac{dz}{z-a}$, or $f^{(n)}(a)$.

Example 2. Verify the same theorem by means of integration by parts.

We have

$$\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = {}_C\left\{-\frac{(n-1)!}{2\pi i}\frac{f(z)}{(z-a)^n}\right\} + \frac{(n-1)!}{2\pi i}\int_C \frac{f'(z)\,dz}{(z-a)^n}$$

and the first term is zero, since $\frac{f(z)}{(z-a)^n}$ resumes its original value when z makes the circuit of the contour C. Proceeding in this way, we have

$$\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{1}{2\pi i}\int_C \frac{f^{(n)}(z)\,dz}{z-a} = f^{(n)}(a).$$

39. *Taylor's Theorem.*

Consider now a function $f(z)$, which is regular in the neighbourhood of a point $z=a$. Let C be the circle of largest radius which can be drawn with a as centre in the z-plane, so as not to include any singular point of the function $f(z)$; so that $f(z)$ is a regular function at all points of C. Let $z=a+h$ be any point within the circle C. Then by § 37, we have

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a-h}$$

$$= \frac{1}{2\pi i}\int_C f(z)\,dz\left\{\frac{1}{z-a} + \frac{h}{(z-a)^2} + \dots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right\}$$

$$= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \frac{1}{2\pi i}\int_C \frac{f(z)\,dz\,.\,h^{n+1}}{(z-a)^{n+1}(z-a-h)}.$$

But at points z on the circle C, the modulus of $\frac{f(z)}{z-a-h}$ will not exceed some finite quantity M.

Therefore

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz\,.\,h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right| \leqslant \frac{M\,.\,2\pi R}{2\pi}\left(\frac{|h|}{R}\right)^{n+1},$$

where R is the radius of the circle C, so that $2\pi R$ is the length of the path of integration in the last integral, and $R=|z-a|$ for points z on the circumference of C.

The right-hand side of the last inequality tends to zero as n increases indefinitely. We have therefore

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \dots,$$

which we can write

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \ldots.$$

This result is known as *Taylor's Theorem*; the proof we have given is due to Cauchy, and shews exactly for what range of values of z the theorem holds true, namely for all points z which are nearer to a than the nearest singularity of $f(z)$. It follows that *the radius of convergence of a power-series is always such as just to exclude from the circle of convergence the nearest singularity of the function represented by the series.*

At this stage we may introduce some terms which will be frequently used.

If $f(a) = 0$, the function $f(z)$ is said to have a *zero* at the point $z = a$. If at such a point $f'(a)$ is different from zero, the zero of $f(a)$ is said to be *simple*; if, on the other hand, the quantities $f'(a), f''(a), \ldots f^{(n-1)}(a)$ are all zero, so that the Taylor's expansion of $f(z)$ at $z = a$ begins with a term in $(z-a)^n$, then the function $f(z)$ is said to have a *zero of the* nth *order* at the point $z = a$.

Example 1. Find a function $f(z)$, which is regular within the circle C of centre at the origin and radius unity, and has the value

$$\frac{a - \cos\theta}{a^2 - 2a\cos\theta + 1} + i\frac{\sin\theta}{a^2 - 2a\cos\theta + 1}$$

(where $a > 1$ and θ is the vectorial angle) at points on the circumference of C.

We have

$$\begin{aligned} f^{(n)}(0) &= \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}} \\ &= \frac{n!}{2\pi i}\int_0^{2\pi} e^{-ni\theta} \cdot i\,d\theta \cdot \frac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1} \qquad \text{putting } z = e^{i\theta} \\ &= \frac{n!}{2\pi}\int_0^{2\pi} \frac{e^{-ni\theta}\,d\theta}{a - e^{i\theta}} = \frac{n!}{2\pi}\sum_{k=0}^{\infty} \frac{1}{a^{k+1}}\int_0^{2\pi} e^{(k-n)i\theta}\,d\theta \\ &= \frac{n!}{a^{n+1}}, \text{ since the only non-zero term is that from } k = n. \end{aligned}$$

Therefore by Maclaurin's Theorem*,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}},$$

or $f(z) = \dfrac{1}{a-z}$ for all points within the circle.

This example raises the interesting question, What is $f(z)$ for points outside the circle? Is it still $\dfrac{1}{a-z}$? This will be discussed in §§ 41, 42.

Example 2. Prove that the arithmetic mean of all values of $z^{-n}\sum_{\nu=0}^{\infty} a_\nu z^\nu$, for points z on the circumference of the circle $|z| = 1$, is a_n, if $\Sigma a_\nu z^\nu$ is regular at all points within the circle.

* The result
$$f(z) = f(0) + zf'(0) + \frac{z^2}{2} f''(0) + \ldots,$$
which is obtained by putting $a = 0$ in Taylor's Theorem, is usually called *Maclaurin's Theorem.*

Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu = f(z)$, so that $a_\nu = \frac{f^{(\nu)}(0)}{\nu!}$. Then the required mean is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(z)\,d\theta}{z^n}, \text{ where } z=e^{i\theta},$$

or

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}}, \text{ where } C \text{ is the circle,}$$

or

$$\frac{f^{(n)}(0)}{n!},$$

or

$$a_n.$$

Example 3. Prove that if h is a given constant, and $(1-2zh+h^2)^{-\frac{1}{2}}$ is expanded in the form

$$1+hP_1(z)+h^2P_2(z)+h^3P_3(z)+\ldots \qquad \text{.........................(A)},$$

where $P_n(z)$ is easily seen to be a polynomial of degree n in z, then this series converges so long as z is in the interior of an ellipse whose foci are the points $z=1$ and $z=-1$, and whose semi-major axis is $\frac{1}{2}\left(h+\frac{1}{h}\right)$.

Let the series be first regarded as a function of h. It is a power-series in h, and therefore converges so long as the point h lies within a circle on the h-plane. The centre of this circle is the point $h=0$, and its circumference will be such as to pass through that singularity of $(1-2zh+h^2)^{-\frac{1}{2}}$ which is nearest to $h=0$.

But
$$1-2zh+h^2=(h-z+\sqrt{z^2-1})\,(h-z-\sqrt{z^2-1}),$$
so the singularities of $(1-2zh+h^2)^{-\frac{1}{2}}$ are the points $h=z-(z^2-1)^{\frac{1}{2}}$ and $h=z+(z^2-1)^{\frac{1}{2}}$, at which it is infinite.

Thus the series (A) converges so long as $|h|$ is less than either

$$|z-(z^2-1)^{\frac{1}{2}}| \text{ or } |z+(z^2-1)^{\frac{1}{2}}|.$$

Now draw an ellipse in the z-plane passing through the point z and having its foci at the points 1 and -1. Let a be its semi-major axis, and θ the eccentric angle of z on it.

Then
$$z=a\cos\theta+i(a^2-1)^{\frac{1}{2}}\sin\theta,$$
which gives
$$z\pm(z^2-1)^{\frac{1}{2}}=\{a\pm(a^2-1)^{\frac{1}{2}}\}(\cos\theta\mp i\sin\theta),$$
so
$$|z\pm(z^2-1)^{\frac{1}{2}}|=a\pm(a^2-1)^{\frac{1}{2}}.$$

Thus the series (A) converges so long as h is less than the least of the quantities $a+(a^2-1)^{\frac{1}{2}}$ and $a-(a^2-1)^{\frac{1}{2}}$, i.e. so long as h is less than $a-(a^2-1)^{\frac{1}{2}}$. But

$$h=a-(a^2-1)^{\frac{1}{2}} \text{ when } a=\frac{1}{2}\left(h+\frac{1}{h}\right).$$

Therefore the series (A) converges so long as z is within an ellipse whose foci are 1 and -1, and whose semi-major axis is $\frac{1}{2}\left(h+\frac{1}{h}\right)$.

40. *Forms of the remainder in Taylor's Series.*

The form found in the last article for the remainder after n terms in Taylor's series is

$$R_n=\frac{1}{2\pi i}\int_C \frac{f(z)\,h^n\,dz}{(z-a)^n(z-a-h)}.$$

It is not difficult to derive from this expression the forms of the remainder usually given in treatises on the Differential and Integral Calculus. For on integrating by parts the quantity $n\int_0^h \frac{(h-t)^{n-1}dt}{(z-a-t)^{n+1}}$, we have

$$n\int_0^h \frac{(h-t)^{n-1}dt}{(z-a-t)^{n+1}} = \frac{h^n}{(z-a)^{n+1}} + (n+1)\int_0^h \frac{(h-t)^n dt}{(z-a-t)^{n+2}}$$

$$= \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+2}} + \dots,$$

by successive repetition of this process,

$$= \frac{h^n}{(z-a)^n (z-a-h)}.$$

So
$$R_n = \frac{n}{2\pi i}\int_0^h (h-t)^{n-1}\,dt \int_C \frac{f(z)\,dz}{(z-a-t)^{n+1}},$$

or
$$R_n = \frac{1}{(n-1)!}\int_0^h (h-t)^{n-1} f^{(n)}(a+t)\,dt,$$

which is a new form for the remainder.

Now suppose that all the quantities concerned are real. Then along the line of integration, $(h-t)^{n-1}$ has a fixed sign, so

$$R_n = \frac{H}{(n-1)!}\int_0^h (h-t)^{n-1}\,dt,$$

where H lies between the greatest and least values of $f^{(n)}(a+t)$ between $t=0$ and $t=h$. We can therefore write $H=f^{(n)}(a+\theta h)$, where $0<\theta<1$, and then

$$R = \frac{1}{(n-1)!} f^{(n)}(a+\theta h)\int_0^h (h-t)^{n-1}\,dt,$$

or
$$R = \frac{h^n}{n!} f^{(n)}(a+\theta h),$$

which is *Lagrange's form for the remainder.*

Darboux gave in 1876 (*Journal de Math.* (3) II. p. 291) a form for the remainder in Taylor's Series, which is applicable to complex variables and resembles the above form given by Lagrange for the case of real variables.

41. *The Process of Continuation.*

Near every point $P(z_0)$ at which a function $f(z)$ is regular, we have seen that there is an expansion for the function as a series of ascending positive integral powers of $(z-z_0)$, the coefficients in which are the successive derivates of the function at z_0.

Now let A be the singularity of $f(z)$ which is nearest to P. Then the circle within which this expansion is valid has P for centre and PA for radius.

Suppose that we are given the values of the function at all points of the circumference of this circle, or more strictly speaking, of a circle slightly smaller than this and concentric with it: then the preceding theorems enable us to find its value at all points *within* the circle. The question arises, How can the values of the function at points *outside* the circle be found?

In other words, *given a power-series which converges and represents a function only at points within a circle, to derive from it the values of the function at points outside the circle.*

For this purpose choose any point P_1 within the circle, not on the line PA. We know the value of the function and all its derivates at P_1, from the series, and so we can form the Taylor series with P_1 as origin, which will represent the function for all points within some circle of centre P_1. Now this circle will extend as far as the singularity which is nearest to P_1, which may or not be A; but in either case, this new circle will generally* lie partly outside the old circle of convergence, and *for points in the region which is included in the new circle but not in the old circle, the new series will furnish the values of the function, although the old series failed to do so.*

Similarly we can take any other point P_2, in the region for which the values of the function are now known, and form the Taylor series with P_2 as origin, which will in general furnish the values of the function for other points at which its values were not previously known; and so on.

This method is called *continuation*†. By means of it, starting from a representation of a function by any one power-series we can find any number of other power-series, which between them furnish the value of the function at all points where it exists; and the aggregate of all the power-series thus obtained constitutes the analytical expression of the function.

Example. The series

$$\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\frac{z^3}{a^4}+\dots$$

represents the function

$$f(z)=\frac{1}{a-z}$$

only for points z within the circle $|z|=a$.

But any number of other power-series exist, of the type

$$\frac{1}{a-b}+\frac{z-b}{(a-b)^2}+\frac{(z-b)^2}{(a-b)^3}+\frac{(z-b)^3}{(a-b)^4}+\dots,$$

which represent the function for points outside this circle.

* The word "generally" must be taken as referring to the cases which are likely to come under the student's notice before he reads the more advanced parts of the subject.

† In German, *Fortsetzung*.

On functions to which the continuation-process cannot be applied.

It is not always possible to carry out the process of continuation. Take as an example the function $f(z)$ defined by the power-series

$$f(z) = 1 + z^2 + z^4 + z^8 + z^{16} + \ldots + z^{2^n} + \ldots,$$

which clearly converges in the interior of a circle whose radius is unity and whose centre is at the origin.

Now as z approaches the value $+1$ by real values, the value of $f(z)$ obviously tends towards $+\infty$; the point $+1$ is therefore a singularity of $f(z)$.

But
$$f(z) = z^2 + f(z^2),$$
so if z is such that $z^2 = 1$, and therefore $f(z^2)$ is infinite, then $f(z)$ is also infinite, and so z is a singularity of $f(z)$: the point $z = -1$ is therefore a singularity of $f(z)$.

Similarly since
$$f(z) = z^2 + z^4 + f(z^4),$$
we see that if z is such that $z^4 = 1$, then z is a singularity of $f(z)$; and in general, any root of any of the equations
$$z^2 = 1, \quad z^4 = 1, \quad z^8 = 1, \quad z^{16} = 1, \quad \ldots,$$
is a singularity of $f(z)$. But these points all lie on the circle $|z| = 1$; and in any arc of this circle, however small, there are an infinite number of them. The attempt to carry out the process of continuation will therefore be frustrated by the existence of this unbroken front of singularities, beyond which it is impossible to pass.

In such a case the function $f(z)$ *does not exist at all* for points z situated outside the circle $|z| = 1$; the circle is said to be a *limiting circle* for the function.

42. *The identity of a function.*

The two series
$$1 + z + z^2 + z^3 + \ldots$$
and
$$-1 + (z-2) - (z-2)^2 + (z-2)^3 - (z-2)^4 + \ldots$$
are not simultaneously convergent for any value of z, and are distinct expansions. Nevertheless, we generally say that they represent *the same function*, on the strength of the fact that they can both be represented by the same rational expression $\dfrac{1}{1-z}$.

This raises the question of the *identity* of a function. Under what circumstances shall we say that two *different* expansions represent the *same* function?

We shall define a function, by means of the last article, as consisting of one power-series together with all the other power-series which can be derived from it by the process of continuation. Two different analytical expressions will therefore be regarded as defining the same function if they represent power-series which can be derived from each other by continuation.

It is important to observe that *a single analytical expression can represent different functions in different parts of the plane.* This can be seen from the following example.

Consider the series

$$\frac{1}{2}\left(z+\frac{1}{z}\right)+\sum_{n=1}^{\infty}\left(z-\frac{1}{z}\right)\left(\frac{1}{1+z^n}-\frac{1}{1+z^{n-1}}\right).$$

The sum of the first n terms of this series is

$$\frac{1}{z}+\left(z-\frac{1}{z}\right)\cdot\frac{1}{1+z^n}.$$

The series therefore converges for all finite values of z. But since when n is infinitely great, z^n is infinitely small or infinitely great according as $|z|$ is less or greater than unity, we see that the sum to infinity of the series is z when $|z|<1$, and $\frac{1}{z}$ when $|z|>1$. *This series therefore represents one function at points in the interior of the circle* $|z|=1$, *and an entirely different function at points outside the same circle.*

Example. Shew that the series

$$z+\frac{z^3}{3^2}+\frac{z^5}{5^2}+\frac{z^7}{7^2}+\dots$$

and

$$\frac{1}{2}\left\{\frac{2z}{1-z^2}-\frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{2z}{1-z^2}\right)^3+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{5}\cdot\left(\frac{2z}{1-z^2}\right)^5-\dots\right\}$$

represent the same function in the common part of their domain of convergence.

43. *Laurent's Theorem.*

A very important extension of Taylor's Theorem was published in 1843 by Laurent; it relates to the expansion of functions under circumstances in which Taylor's Theorem cannot be applied.

Let C and C' be two concentric circles of centre a, of which C' is the inner; and let $f(z)$ be a function which is regular at all points in the ring-shaped space between C and C'. Let $a+h$ be any point in this ring-shaped space. Then we have (§ 37, Corollary)

$$f(a+h)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a-h}\,dz-\frac{1}{2\pi i}\int_{C'}\frac{f(z)}{z-a-h}\,dz,$$

where the integrals are supposed taken in the positive or counter-clockwise direction round the circles.

This can be written

$$f(a+h)=\frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{z-a}+\frac{h}{(z-a)^2}+\dots+\frac{h^n}{(z-a)^{n+1}}+\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right\}dz$$

$$+\frac{1}{2\pi i}\int_{C'}f(z)\left\{\frac{1}{h}+\frac{z-a}{h^2}+\dots+\frac{(z-a)^n}{h^{n+1}}+\frac{(z-a)^{n+1}}{h^{n+1}(z-a-h)}\right\}dz.$$

We find, as in the proof of Taylor's Theorem, that

$$\int_C \frac{f(z)\,dz\,.\,h^{n+1}}{(z-a)^{n+1}(z-a-h)} \text{ and } \int_{C'} \frac{f(z)\,dz\,(z-a)^{n+1}}{(z-a-h)\,h^{n+1}}$$

tend to zero as n increases indefinitely; and thus we have

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \ldots + \frac{b_1}{h} + \frac{b_2}{h^2} + \ldots,$$

where $\quad a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C'} (z-a)^{n-1} f(z)\,dz.$

This result is *Laurent's Theorem*; changing the notation, it can be expressed in the following form: *If z be any point in the ring-shaped space within which $f(z)$ is regular, and which is bounded by the two concentric circles C and C' of centre a, then $f(z)$ can be expanded at the point z in the form*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots,$$

where $\quad a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)\,dt}{(t-a)^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C'} (t-a)^{n-1} f(t)\,dt.$

An important case of Laurent's Theorem arises when there is only one singularity within the inner circle C', namely at the centre a. In this case the circle C' can be taken to be infinitely small, and so Laurent's expansion is valid for all points in the interior of the circle C, except the centre a.

Example 1. Prove that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0(x) + zJ_1(x) + z^2 J_2(x) + \ldots + z^n J_n(x) + \ldots$$
$$- \frac{1}{z} J_1(x) + \frac{1}{z^2} J_2(x) - \ldots + \frac{(-1)^n}{z^n} J_n(x) + \ldots,$$

where $$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\,d\theta.$$

For Laurent's Theorem gives

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = a_0 + a_1 z + a_2 z^2 + \ldots + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots,$$

where $\quad a_n = \dfrac{1}{2\pi i}\displaystyle\int_C e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} \frac{dz}{z^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C'} e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} z^{n-1}\,dz,$

and where C and C' are any circles with the origin as centre. Taking C to be the circle of radius unity, and writing $z = e^{i\theta}$, we have

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{ix\sin\theta}\,.\,e^{-ni\theta}\,i\,.\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\,d\theta,$$

since the parts of $\int_0^{2\pi} \sin(n\theta - x\sin\theta)\,d\theta$ which arise from θ and $2\pi-\theta$ will destroy each other. Thus

$$a_n = J_n(x).$$

Now $b_n = (-1)^n a_n$, since the function expanded is unaltered if $-\frac{1}{z}$ be written for z. Thus

$$b_n = (-1)^n J_n(x),$$

which completes the proof.

Example 2. Shew that, in the annulus defined by

$$|a| < |z| < |b|,$$

the expression $\left\{\frac{bz}{(z-a)(b-z)}\right\}^{\frac{1}{2}}$ can be expanded in the form

$$S_0 + \sum_{n=1}^{\infty} S_n\left(\frac{a^n}{z^n} + \frac{z^n}{b^n}\right),$$

where
$$S_n = \sum_{l=0}^{\infty} \frac{1\,.\,3\ldots(2l-1)\,.\,1\,.\,3\ldots(2l+2n-1)}{2^{2l+n}\,.\,l!\,(l+n)!}\left(\frac{a}{b}\right)^l.$$

For by Laurent's Theorem if C denote the circle $|z|=r$, where $|a|<r<|b|$, then the coefficient of z^n in the required expansion is

$$\frac{1}{2\pi i}\int_C \frac{dz}{z^{n+1}}\left\{\frac{bz}{(z-a)(b-z)}\right\}^{\frac{1}{2}}.$$

Putting $z = re^{i\theta}$, this becomes

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta} r^{-n}\,d\theta \left(1 - \frac{r}{b}e^{i\theta}\right)^{-\frac{1}{2}}\left(1 - \frac{a}{r}e^{-i\theta}\right)^{-\frac{1}{2}},$$

or
$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta} r^{-n}\,d\theta \sum_k \frac{1\,.\,3\ldots(2k-1)}{2^k\,.\,k!}\frac{r^k e^{ki\theta}}{b^k}\sum_l \frac{1\,.\,3\ldots(2l-1)}{2^l\,.\,l!}\frac{a^l e^{-li\theta}}{r^l}.$$

The only terms which give integrals different from zero are those arising from $k=l+n$. So the coefficient of z^n is

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \sum_l \frac{1\,.\,3\ldots(2l-1)}{2^l\,.\,l!}\,\frac{1\,.\,3\ldots(2l+2n-1)}{2^{l+n}\,.\,(l+n)!}\,\frac{a^l}{b^{l+n}},$$

or
$$\frac{S_n}{b^n}.$$

Similarly it can be shewn that the coefficient of $\frac{1}{z^n}$ is $S_n a^n$.

Example 3. Shew that

$$e^{uz+\frac{v}{z}} = a_0 + a_1 z + a_2 z^2 + \ldots + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots,$$

where
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\cos\{(u-v)\sin\theta - n\theta\}\,d\theta,$$

and
$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\cos\{(v-u)\sin\theta - n\theta\}\,d\theta.$$

44. *The nature of the singularities of a one-valued function.*

Consider now a function $f(z)$ which is regular at all points of a certain region in the z-plane, except a point $z=a$; so that the point a is a singularity of the function $f(z)$.

Surround the point a by a small circle γ, with a as centre. Then in the ring-shaped space between γ and some larger concentric circle C, the function $f(z)$ can by Laurent's Theorem be expanded in the form

$$A_0 + A_1(z-a) + A_2(z-a)^2 + A_3(z-a)^3 + \dots$$
$$+ \frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \frac{B_3}{(z-a)^3} + \dots.$$

The terms in the last line are called the *Principal Part* of the expansion of the function at the singularity a; if they were non-existent, the function would clearly be regular at the point; so they may be regarded as constituting the analytical expression of the singularity.

Now these terms of the Principal Part may be unlimited in number, i.e. the series

$$\frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \frac{B_3}{(z-a)^3} + \dots$$

may be an infinite series; in this case the point a is said to be an *essential singularity** of the function $f(z)$. Or on the other hand, they may be limited in number, i.e. the series just written down may be a terminating series; so that the expansion can be written in the form

$$\frac{B_n}{(z-a)^n} + \frac{B_{n-1}}{(z-a)^{n-1}} + \dots + \frac{B_1}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \dots.$$

In this case the function is said to have a *pole of order* n at the point a. When n is unity, so that the expansion is of the form

$$\frac{B_1}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \dots,$$

the singularity is said to be a *simple pole.*

Example 1. Find the singularities of the function

$$\frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}}-1}.$$

Near $z=0$, the function can be expanded in the form

$$\frac{e^{-\frac{c}{a}-\frac{cz}{a^2}-\frac{cz^2}{a^3}-\dots}}{\frac{z}{a}+\frac{z^2}{2!\,a^2}+\frac{z^3}{3!\,a^3}+\dots},$$

* The name *essential singularity* is also applied to any singularity of a one-valued function which is not a pole, i.e. to singularities for which no Laurent expansion at all can be found.

or
$$\frac{e^{-\frac{c}{a}}\cdot a}{z} - e^{-\frac{c}{a}}\left(\frac{c}{a}+\frac{1}{2}\right) + \text{positive powers of } z.$$

There is therefore a simple pole at $z=0$. Similarly there is a simple pole at each of the points $2\pi nia$ $(n=\pm 1, \pm 2, \pm 3, \ldots)$.

Near $z=a$, the function can be expanded in the form

$$\frac{e^{\frac{c}{z-a}}}{e \cdot e^{\frac{z-a}{a}} - 1},$$

or
$$\frac{1+\dfrac{c}{z-a}+\dfrac{c^2}{2!\,(z-a)^2}+\ldots}{e\left(1+\dfrac{z-a}{a}+\ldots\right)-1},$$

which gives an expansion involving all positive and negative powers of $(z-a)$. So there is an essential singularity at $z=a$.

There is also an essential singularity at $z=\infty$, as will be seen after the explanations of the next article.

Example 2. Shew that the function defined by the series

$$\sum_{n=1}^{\infty} \frac{nz^{n-1}\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}}{(z^n-1)\left\{z^n-\left(1+\frac{1}{n}\right)^n\right\}}$$

has a simple pole at each of the points

$$z=\left(1+\frac{1}{n}\right)e^{\frac{2ki\pi}{n}} \quad (k=0, 1, 2, \ldots n-1;\ n=1, 2, \ldots \infty).$$

(Cambridge Mathematical Tripos, Part II., 1899.)

45. *The point at infinity.*

The behaviour of a function $f(z)$ for infinite values of the variable z can be brought into consideration in the same way as its behaviour for finite values of z.

For write $z=\dfrac{1}{z'}$, so that the infinite values of z are represented by the point $z'=0$ in the z'-plane. Let $f(z)=\phi(z')$. Then the function $\phi(z')$ may have a zero of order m at the point $z'=0$; in this case the Taylor expansion of $\phi(z')$ will be of the form

$$Az'^m + Bz'^{m+1} + Cz'^{m+2} + \ldots,$$

and so the expansion of $f(z)$ valid near $z=\infty$ will be of the form

$$f(z) = \frac{A}{z^m} + \frac{B}{z^{m+1}} + \frac{C}{z^{m+2}} + \ldots.$$

In this case, $f(z)$ is said to have a *zero of order m at $z=\infty$*.

Again, the function $\phi(z')$ may have a pole of order m at the point $z'=0$; in this case,

$$\phi(z')=\frac{A}{z'^m}+\frac{B}{z'^{m-1}}+\frac{C}{z'^{m-2}}+\ldots+\frac{L}{z'}+M+Nz'+Pz'^2+\ldots,$$

and so for large values of z, $f(z)$ can be expanded in the form

$$f(z)=Az^m+Bz^{m-1}+Cz^{m-2}+\ldots+Lz+M+\frac{N}{z}+\frac{P}{z^2}+\ldots.$$

In this case, $z=\infty$ is said to be a *pole of order m* for the function $f(z)$.

Similarly $f(z)$ is said to have an *essential singularity* at $z=\infty$, if $\phi(z')$ has an essential singularity at the point $z'=0$. Thus the function e^z has an essential singularity at $z=\infty$, since the function $e^{\frac{1}{z'}}$ or

$$1+\frac{1}{z'}+\frac{1}{2z'^2}+\frac{1}{3z'^3}+\ldots$$

has an essential singularity at $z'=0$.

Example. Discuss the function represented by the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{1+a^{2n}z^2}, \qquad (a>1).$$

The function represented by this series has singularities at $z=\frac{i}{a^n}$ and $z=-\frac{i}{a^n}$, $(n=1, 2, 3, \ldots)$, since at each of these points the denominator of one of the terms in the series is zero. These singularities are on the imaginary axis, and are infinitely numerous near the origin $z=0$: so no Taylor or Laurent expansion can be formed for the function valid in the region immediately surrounding the origin.

For values of z other than these singularities, the series converges absolutely, since the ratio of the $(n+1)$th term to the nth is ultimately $\frac{1}{(n+1)a^2}$, which is very small when n is large. The function is an even function of z (i.e. is unchanged if the sign of z be changed), is zero for all infinite values of z, and is regular at all points outside a circle C of radius unity and centre at the origin. So for points outside this circle it can be expanded in the form

$$\frac{b_2}{z^2}+\frac{b_4}{z^4}+\frac{b_6}{z^6}+\ldots,$$

where, by Laurent's Theorem,

$$b_{2k}=\frac{1}{2\pi i}\int_C z^{2k-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{a^{-2n}}{a^{-2n}+z^2}dz.$$

Now $$\frac{a^{-2n}z^{2k-1}}{n!(a^{-2n}+z^2)}=\frac{z^{2k-3}a^{-2n}}{n!}\left(1-\frac{a^{-2n}}{z^2}+\frac{a^{-4n}}{z^4}-\frac{a^{-6n}}{z^6}+\ldots\right),$$

and the coefficient of $\frac{1}{z}$ on the right-hand side of this equation is $\frac{(-1)^{k-1}a^{-2kn}}{n!}$.

Therefore, since only terms in $\frac{1}{z}$ can furnish non-zero integrals, we have

$$b_{2k} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{(-1)^{k-1} a^{-2kn}}{n!} \frac{dz}{z}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{k-1}}{n!\, a^{2kn}}$$

$$= (-1)^{k-1} e^{\frac{1}{a^{2k}}}.$$

Therefore for large values of z (and indeed for all points z outside the circle of radius unity) the function can be expanded in the form

$$\frac{e^{\frac{1}{a^2}}}{z^2} - \frac{e^{\frac{1}{a^4}}}{z^4} + \frac{e^{\frac{1}{a^6}}}{z^6} - \dots.$$

The function has a zero of the second order at $z = \infty$, since the expansion begins with a term in $\frac{1}{z^2}$.

46. *Many-valued functions.*

In all our previous work we have supposed the function $f(z)$ to have one definite value corresponding to each value of z.

But functions exist which have more than one value corresponding to each value of z. Thus the function $z^{\frac{1}{2}}$ has two values (viz. $+\sqrt{z}$ and $-\sqrt{z}$) corresponding to each value of z, and the function $\tan^{-1} z$ has an infinite number of values, expressed by the formula $\tan^{-1} z \pm k\pi$, where k is any integer.

We may however for many purposes consider $+\sqrt{z}$ and $-\sqrt{z}$ as if they were two distinct functions, and apply to either of them separately the theorems which have been investigated in this chapter. When we in this way select some one determination of a many-valued function for consideration, it is called a *branch* of the many-valued function. Thus the values $\log z$, $\log z + 2\pi i$, $\log z + 4\pi i, \dots$, would be said to belong to different branches of the function $\log z$.

There will be certain points for which the values of the function given by different branches coincide: these points are called *branch-points* of the function, and must be included among its singularities. Thus the function $z^{\frac{1}{2}}$ has a branch-point at $z = 0$, since either branch there gives the same value, zero, for the function.

It must not however be supposed that the branches of a many-valued function really are distinct functions. The following example shews how the different branches of a many-valued function change into each other.

Let $$f(z) = z^{\frac{1}{2}}.$$

Write $z = r(\cos\theta + i\sin\theta)$, where $0 < \theta < 2\pi$. Then the two values of $f(z)$ are

$$+\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \text{ and } -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Let us take the former of these values, and consider its changes as the point z describes a circle round the origin ($z = 0$). As the point travels, r is unchanged, but θ constantly increases, and when the point reaches again the starting-point after completing the circuit, θ has increased by 2π. The function has therefore become

$$+\sqrt{r}\left(\cos\frac{\theta+2\pi}{2} + i\sin\frac{\theta+2\pi}{2}\right),$$

or

$$-\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

In other words, *the branch of the function with which we started has passed over into the other branch.*

In following the succession of values of $f(z)$ along a given path, the final value is deduced without ambiguity from the initial value; and all conceivable paths lead to one of two final values, viz. $\sqrt{z}$ and $-\sqrt{z}$. But it appears from the above that it is not possible to keep these branches permanently apart as distinct functions, because paths lead from one value to the other.

The idea of the different *branches* of a function helps us to understand many of the "paradoxes" of mathematics, such as the following.

Consider the function

$$u = z^z,$$

for which

$$\frac{du}{dz} = z^z(1 + \log z).$$

When z is negative and real, $\frac{du}{dz}$ is not real. Now if z is a negative quantity of the form $\frac{p}{2q+1}$ (where p and q are positive or negative integers), u is real.

If therefore we draw the real curve

$$u = z^z,$$

we have for negative values of z a series of conjugate points, arranged at infinitely small intervals of z: and thus we may think of proceeding to form the tangent as the limit of the chord, just as if the curve were continuous; and thus $\frac{du}{dz}$, when derived from the inclination of the tangent to the axis of x, would appear to be real. The question thus arises, Why does the ordinary process of differentiation give a non-real value for $\frac{du}{dz}$? The

explanation is, that these conjugate points do not all arise from the same *branch* of the function $u=z^z$. We have in fact

$$u=e^{z \log z},$$

and $\log z$ has an arbitrary additive part $2k\pi i$, where k is any integer. To each value of k corresponds one branch of the function u. Now in order to get a real value of u when z is negative, we have to choose a suitable value for k: and *this value of k varies as we go from one conjugate point to an adjacent one.* So the conjugate points do not represent values of u arising from the same branch of the function $u=z^z$, and consequently we cannot expect the value of $\frac{du}{dz}$ to be given by the tangent of the inclination to the axis of x of the tangent-line to the series of conjugate points.

Example 1. If $\log z$ be defined by the equation

$$\log z = \underset{n=\infty}{\text{Limit}}\, n\,(z^{\frac{1}{n}}-1),$$

shew that $\log z$ is a many-valued function, which increases by $2\pi i$ when z describes a closed path round the origin.

For put $$z=r\,(\cos\theta+i\sin\theta).$$

Then one of the values of $\log z$, on this definition, is

$$\underset{n=\infty}{\text{Limit}}\, n\left\{r^{\frac{1}{n}}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)-1\right\},$$

where $r^{\frac{1}{n}}$ is the positive nth root of r.

This can be written

$$\underset{n=\infty}{\text{Limit}}\, n\,\{r^{\frac{1}{n}}-1\}+i\theta.$$

When z describes a closed path round the origin, the first term in this expression remains unaltered, while the second increases by $2\pi i$; hence the result.

Example 2. Find the points at which the following functions are not regular.

(*a*) z^2. Answer, $z=\infty$.

(*b*) $\operatorname{cosec} z$. Answer, $z=0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

(*c*) $\dfrac{z-1}{z^2-5z+6}$. Answer, $z=2, 3$.

(*d*) $e^{\frac{1}{z}}$. Answer, $z=0$.

(*e*) $\{(z-1)\,z\}^{\frac{1}{3}}$. Answer, $z=0, 1, \infty$.

Example 3. Prove that if the different values of a^z, corresponding to a given value of z, are represented on an Argand diagram, the representative points will be the vertices of an equiangular polygon inscribed in an equiangular spiral, the angle of the spiral being independent of a.

(Cambridge Mathematical Tripos, Part I., 1899.)

47. *Liouville's Theorem.*

We know by § 38 that if $f(z)$ be any function of z which is regular at all points of the z-plane within a circle C, of centre a and radius r, then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Now let M be the greatest value of $|f(z)|$ at points on the circle C. Then this equation gives (§ 32)

$$|f^{(n)}(a)| \leqslant \frac{n!}{2\pi}\frac{M}{r^{n+1}}\cdot 2\pi r$$

$$\leqslant \frac{n!\,M}{r^n}.$$

From this inequality an important consequence can be deduced. Suppose that $f(z)$ is, if possible, a regular function of z *over the whole z-plane, including infinity*, i.e. that it has no singularities at all.

Then in the above equation M is finite when r is infinite, whatever n is; and therefore $f^{(n)}(a)$ is zero for all values of n and a, i.e. $f(a)$ is a constant independent of a. We thus arrive at *Liouville's theorem*, that *the only function which is regular everywhere is a constant*.

As will be seen in the next article, and again frequently in the latter half of this volume, Liouville's theorem furnishes short and convenient proofs for some of the most important results in Analysis.

48. *Functions with no essential singularities.*

We shall now shew that *the only one-valued functions which have no singularities in either the finite or infinite part of the plane, except poles, are rational functions.*

For let $f(z)$ be such a function; let its singularities in the finite part of the plane be at the points $c_1, c_2, \ldots c_k$: and let the principal part (§ 44) of its expansion at the pole c_r be

$$\frac{a_{r,1}}{z-c_r} + \frac{a_{r,2}}{(z-c_r)^2} + \ldots + \frac{a_{r,n_r}}{(z-c_r)^{n_r}}.$$

Let the principal part of its expansion at the pole $z=\infty$ be

$$a_1 z + a_2 z^2 + \ldots + a_n z^n;$$

if $z=\infty$ is not a pole, but a regular point for the function, then the coefficients in this expansion will be zero.

Now the function

$$f(z) - \sum_{r=1}^{k}\left\{\frac{a_{r,1}}{z-c_r} + \frac{a_{r,2}}{(z-c_r)^2} + \ldots + \frac{a_{r,n_r}}{(z-c_r)^{n_r}}\right\}$$
$$- a_1 z - a_2 z^2 - \ldots - a_n z^n$$

has clearly no singularities at the points $c_1, c_2, \ldots c_k, \infty$; it has therefore no singularities at all, and so by Liouville's theorem is a constant; that is,

$$f(z) = \text{constant} + a_1 z + a_2 z^2 + \ldots + a_n z^n$$
$$+ \sum_{r=1}^{k} \left\{ \frac{a_{r,1}}{z - c_r} + \frac{a_{r,2}}{(z - c_r)^2} + \ldots + \frac{a_{r,n_r}}{(z - c_r)^{n_r}} \right\};$$

$f(z)$ is therefore a rational function, and the theorem is established.

Miscellaneous Examples.

1. Obtain the expansion

$$f(z) = f(a) + 2\left\{\frac{z-a}{2} f'\left(\frac{z+a}{2}\right) + \frac{(z-a)^3}{2^3 . 3!} f'''\left(\frac{z+a}{2}\right) + \frac{(z-a)^5}{2^5 . 5!} f^{(5)}\left(\frac{z+a}{2}\right) + \ldots\right\}.$$

2. Obtain the expansion

$$f(z) = f(a) + \frac{z-a}{2m}\left[f'(a) + f'(z) + 2\left\{f'\left(a + \frac{z-a}{m}\right) + f'\left(a + \frac{2z-2a}{m}\right) + \ldots + f'\left(a + \frac{(m-1)(z-a)}{m}\right)\right\}\right]$$
$$+ \frac{(z-a)^2}{2^2 . 2!\, m^2}\{f''(a) - f''(z)\}$$
$$+ \frac{(z-a)^3}{2^3 . 3!\, m^3}\left[f'''(a) + f'''(z) + 2\left\{f'''\left(a + \frac{z-a}{m}\right) + f'''\left(a + \frac{2z-2a}{m}\right) + \ldots + f'''\left(a + \frac{(m-1)(z-a)}{m}\right)\right\}\right]$$
$$+ \frac{(z-a)^4}{2^4 . 4!\, m^4}\{f^{(4)}(a) - f^{(4)}(z)\}$$
$$+ \ldots$$

(Corey.)

3. Obtain the expansion

$$f(z) = f(a) + \frac{z-a}{m}\left[f'\left(a + \frac{z-a}{2m}\right) + f'\left\{a + \frac{3(z-a)}{2m}\right\} + \ldots + f'\left\{a + \frac{(2m-1)(z-a)}{2m}\right\}\right]$$
$$+ \frac{2}{3!}\left(\frac{z-a}{2m}\right)^3\left[f'''\left(a + \frac{z-a}{2m}\right) + f'''\left\{a + \frac{3(z-a)}{2m}\right\} + \ldots + f'''\left\{a + \frac{(2m-1)(z-a)}{2m}\right\}\right]$$
$$+ \frac{2}{5!}\left(\frac{z-a}{2m}\right)^5\left[f^{(5)}\left(a + \frac{z-a}{2m}\right) + f^{(5)}\left\{a + \frac{3(z-a)}{2m}\right\} + \ldots + f^{(5)}\left\{a + \frac{(2m-1)(z-a)}{2m}\right\}\right]$$
$$+ \ldots$$

(Corey.)

4. In order that values $U + Vi$, which are given as continuous functions of the arc of a circle, should be the boundary values of an analytic function, shew that it is necessary and sufficient:

(a) That $\dfrac{U(a-\psi) - U(a+\psi)}{\psi}$ at the place $\psi = 0$ should be uniformly integrable for all values of a;

(b) That the values of V shall be given by

$$V(a) = \frac{1}{2\pi}\int_0^{\pi}\{U(a-\psi) - U(a+\psi)\}\cot\tfrac{1}{2}\psi\, d\psi.$$

(Tauber.)

5. Shew that for the series

$$\sum_{n=0}^{\infty} \frac{1}{z^n + z^{-n}},$$

the region of convergence consists of two distinct areas, namely outside and inside a circle of radius unity, and that in each of these the series represents one function and represents it completely.

(Weierstrass.)

6. Shew that

$$(1-z^2)^{-\frac{1}{2}} = 1 + \frac{1}{2} z^2 + \frac{1 \cdot 3}{2 \cdot 4} z^4 + \ldots + \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} z^{2n}$$
$$+ \frac{3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \ldots (2n)} (1-z^2)^{-\frac{1}{2}} \int_0^z t^{2n+1} (1-t^2)^{-\frac{1}{2}} \, dt.$$

(Jacobi & Scheibner.)

7. Shew that

$$(1-z)^{-m} = 1 + \frac{m}{1} z + \frac{m(m+1)}{2\,!} z^2 + \ldots + \frac{m(m+1)\ldots(m+n-1)}{n\,!} z^n$$
$$+ \frac{m(m+1)\ldots(m+n)}{n\,!} (1-z)^{-m} \int_0^z t^n (1-t)^{m-1} \, dt.$$

(Jacobi & Scheibner.)

8. Shew that

$$(1-z^2)^{-\frac{1}{2}} \int_0^z t^m (1-t^2)^{-\frac{1}{2}} \, dt = \frac{z^{m+1}}{m+1} \left\{ 1 + \frac{m+2}{m+3} z^2 + \ldots + \frac{(m+2)\ldots(m+2n-2)}{(m+3)\ldots(m+2n-1)} z^{2n-2} \right\}$$
$$+ (1-z^2)^{-\frac{1}{2}} \frac{(m+2)(m+4)\ldots(m+2n)}{(m+1)(m+3)\ldots(m+2n-1)} \int_0^z t^{m+2n} (1-t^2)^{-\frac{1}{2}} \, dt.$$

(Jacobi & Scheibner.)

9. If, in the expansion of $(a + a_1 z + a_2 z^2)^m$ by the multinomial theorem, the remainder after n terms be denoted by R, so that

$$(a + a_1 z + a_2 z^2)^m = A_0 + A_1 z + A_2 z^2 + \ldots + A_{n-1} z^{n-1} + R,$$

shew that

$$R = (a + a_1 z + a_2 z^2)^m \int_0^z \frac{na\, A_n t^{n-1} + (2m-n+1)\, a_2 A_{n-1} t^n}{(a + a_1 t + a_2 t^2)^{m+1}} \, dt.$$

(Jacobi & Scheibner.)

10. If
$$(a_0 + a_1 z + a_2 z^2)^{-m-1} \int_0^z (a_0 + a_1 t + a_2 t^2)^m \, dt$$

be expanded in ascending powers of z in the form

$$A_1 z + A_2 z^2 + \ldots,$$

shew that the remainder after $(n-1)$ terms is

$$(a_0 + a_1 z + a_2 z^2)^{-m-1} \int_0^z (a_0 + a_1 t + a_2 t^2)^m \{n a_0 A_n - (2m+n+1)\, a_2 A_{n-1} t\} \, t^{n-1} \, dt.$$

(Jacobi & Scheibner.)

11. Shew that the series

$$\sum_{n=0}^{\infty} \{1 + \lambda_n(z)\, e^z\} \frac{d^n \phi(z)}{dz^n},$$

where
$$\lambda_n(z) = -1 + z - \frac{z^2}{2\,!} + \frac{z^3}{3\,!} - \ldots \pm \frac{z^n}{n\,!},$$

and where $\phi(z)$ is a regular function of z near $z=0$, is convergent in the neighbourhood of the point $z=0$; and shew that if the sum of the series be denoted by $f(z)$, then $f(z)$ satisfies the differential equation

$$f'(z)=f(z)-\phi(z). \qquad \text{(Pincherle.)}$$

12. Shew that the arithmetic mean of the squares of the moduli of all the values of the series $\sum\limits_0^\infty a_k z^k$ on a circle $|z|=r$, situated within its circle of convergence, is equal to the sum of the squares of the moduli of the separate terms.

(Gutzmer.)

13. Shew that the series

$$\sum_{m=1}^{\infty} e^{-2(am)^{\frac{1}{2}}} z^{m-1}$$

converges when $|z|<1$; and that the function which it represents can also be represented when $|z|<1$ by the integral

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cdot \frac{e^{-\frac{a}{x}}}{e^x - z} \frac{dx}{x^{\frac{3}{2}}},$$

and that it has no singularities except at the point $z=1$. (Lerch.)

14. Shew that the series

$$\frac{2}{\pi}(z+z^{-1})+\frac{2}{\pi}\Sigma\left\{\frac{z}{(1-2\nu-2\nu' zi)(2\nu+2\nu' zi)^2}+\frac{z^{-1}}{(1-2\nu-2\nu' z^{-1}i)(2\nu+2\nu' z^{-1}i)^2}\right\},$$

in which the summation extends over all integral values of ν, ν', except the combination ($\nu=0$, $\nu'=0$), converges absolutely for all values of z except purely imaginary values; and that its sum is $+1$ or -1, according as the real part of z is positive or negative.

(Weierstrass.)

15. Shew that $\sin\left\{u\left(z+\frac{1}{z}\right)\right\}$ can be expanded in a series of the type

$$a_0+a_1 z+a_2 z^2+\ldots+\frac{b_1}{z}+\frac{b_2}{z^2}+\ldots,$$

in which the coefficient of either z^n or z^{-n} is

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(2u\cos\theta)\cos n\theta\, d\theta.$$

16. If

$$f(z)=\sum_{n=1}^{\infty}\frac{z^2}{n^2 z^2+a^2},$$

hew that $f(z)$ is finite and continuous for all real values of z, but cannot be expanded as Maclaurin's series in ascending powers of z; and explain this apparent anomaly.

CHAPTER IV.

THE UNIFORM CONVERGENCE OF INFINITE SERIES.

49. *Uniform Convergence.*

We have seen* that the sum of a convergent series of analytic functions of a variable z can have discontinuities as z varies. It was found by Stokes† and Seidel‡ in 1848 that this can never happen except in association with another phenomenon, that of *non-uniform convergence,* which will now be investigated.

Consider the series

$$S = \frac{z^2}{(1+2z)(1+2z+z^2)} + \frac{z+z^2(z-1)}{(1+2z+z^2)(1+3z+z^3)} + \dots$$
$$+ \frac{z+z^n(z-1)}{(1+nz+z^n)\{1+(n+1)z+z^{n+1}\}} + \dots.$$

We shall first shew that this series is convergent for all values of z except certain isolated points.

For, except for the roots of $1+nz+z^n=0$, the nth term can be put in the form

$$\frac{1}{1+nz+z^n} - \frac{1}{1+(n+1)z+z^{n+1}};$$

so the sum of the first n terms is

$$S_n = \frac{1}{1+2z} - \frac{1}{1+(n+1)z+z^{n+1}},$$

which, as n becomes infinitely great, tends to the value $\dfrac{1}{1+2z}$ for all points except $z=0$: and for $z=0$, we have $S=0$.

Thus (except at the roots of the equations $1+nz+z^n=0$) the series converges; and it represents a regular function, except at $z=0$, where it has a discontinuity.

* In § 42.

† *Collected Papers*, Vol. I. p. 236.

‡ *Münch. Abh.*

What lies at the root of the discontinuity?

The remainder after n terms is

$$R_n = \frac{1}{1 + (n+1)z + z^{n+1}}.$$

For ordinary values of z, say $z = 1$, this remainder decreases rapidly as n increases. Thus if $n = 10$, $z = 3$, the remainder $= \frac{1}{34 + 3^{11}}$, a negligible quantity. But now let z approach near to its discontinuity 0: say $z = \frac{1}{1000000}$. Then with this value of z, the remainder after 1000 terms is nearly 1, and the remainder after 1000000 terms is still nearly $\frac{1}{2}$. This shews that, *as z approaches the discontinuity at $z = 0$, the terms which contribute sensibly to the sum tend to recede to the infinitely distant part of the series, so the first* 1000 *terms do not furnish a good approximation at all.*

We can express this analytically as follows:—The number of terms n, which we have to take in order to make $|R_n|$ less than a given small positive quantity ϵ, tends to ∞ as we approach the point of discontinuity.

This circumstance is the basis of the following definition:—

Let $$u_1(z) + u_2(z) + u_3(z) + u_4(z) + \dots$$

be a series of functions of z, which is convergent at all points z within a given area in the z-plane. Let R_n be the remainder after n terms. Then since the series converges, if we take a small finite quantity ϵ we can find at any point on the area a number r (varying from point to point) such that $|R_n| < \epsilon$ so long as $n > r$. If the numbers r corresponding to the aggregate of points in the vicinity of a given point z are all less than some definite finite number, the series is said to be *uniformly convergent* at the point z; but if near any point z the number r tends to infinity, so that no definite upper limit can be assigned to it, the convergence of the series is said to be non-uniform* in the neighbourhood of the point z.

Example 1. Shew that the series

$$z^2 + \frac{z^2}{1+z^2} + \frac{z^2}{(1+z^2)^2} + \dots + \frac{z^2}{(1+z^2)^n} + \dots,$$

which converges absolutely for all real values of z, is discontinuous at $z=0$ and is non-uniformly convergent in the neighbourhood of $z=0$.

The sum of the first n terms is easily seen to be $1 + z^2 - \frac{1}{(1+z^2)^{n-1}}$. So when z is not zero the sum is $1+z^2$, and when z is zero the sum is zero.

* An interesting geometrical treatment of uniform convergence is given by Osgood in Vol. III. of the *Bull. of the Amer. Math. Soc.* p. 59 (1896).

The remainder after n terms is $\dfrac{1}{(1+z^2)^{n-1}}$. This can be made smaller than any assigned small finite positive quantity ϵ by choosing n so that $n-1 > \dfrac{\log\frac{1}{\epsilon}}{\log(1+z^2)}$. But as z tends to zero, $\dfrac{1}{\log(1+z^2)}$ tends to infinity, so n must tend to infinity, i.e. we have to include an infinite number of terms in order to get the remainder less than ϵ. This series is therefore non-uniformly convergent in the neighbourhood of $z=0$.

Example 2. Shew that at $z=0$ the sum of the series

$$\frac{z}{1(z+1)} + \frac{z}{(z+1)(2z+1)} + \dots + \frac{z}{\{(n-1)z+1\}\{nz+1\}} + \dots$$

is discontinuous and the series is non-uniformly convergent.

The sum of the first n terms is easily seen to be $1 - \dfrac{1}{nz+1}$; so when z is zero the sum is 0.

The remainder after n terms of the series is $\dfrac{1}{nz+1}$; so when z is nearly zero, say $z=$ one-hundred-millionth, the remainder after a million terms is $\dfrac{1}{\frac{1}{100}+1}$ or $1-\dfrac{1}{101}$, so the first million terms of the series do not contribute one per cent. of the sum. And in general if z be small, it is necessary to take n large compared with the large quantity $\dfrac{1}{z}$ in order to make the remainder after n terms small. There is therefore non-uniform convergence in the neighbourhood of $z=0$.

Example 3. Discuss the series

$$\sum_{n=1}^{\infty} \frac{z\{n(n+1)z^2-1\}}{\{1+n^2z^2\}\{1+(n+1)^2z^2\}}.$$

The nth term can be written $\dfrac{nz}{1+n^2z^2} - \dfrac{(n+1)z}{1+(n+1)^2z^2}$, so the sum to infinity is $\dfrac{z}{1+z^2}$, and the remainder after n terms is $\dfrac{(n+1)z}{1+(n+1)^2z^2}$.

However great n may be, if we take z equal to $\dfrac{1}{n+1}$, this remainder will have a finite value, namely $\frac{1}{2}$; the series is therefore non-uniformly convergent at $z=0$.

NOTE. In this example the sum of the series is not discontinuous at $z=0$.

Cayley* regards non-uniform convergence as consisting essentially in the occurrence of a discontinuity in the sum of a series. The condition for a discontinuity in a series

$$u_1(z) + u_2(z) + u_3(z) + \dots$$

at the point $z=a$ is that the series

$$\sum_{r=1}^{\infty} \frac{u_r(a) - u_r(z)}{a-z}$$

shall have an indefinitely large sum when $(a-z)$ is indefinitely small.

* "Note on Uniform Convergence," *Proc. Roy. Soc. Edinb.* XIX. (1891–2), pp. 203–8.

Thus in the series

$$(1-z)+z(1-z)+z^2(1-z)+\ldots,$$

which is non-uniformly convergent and discontinuous at $z=1$, we have

$$\frac{u_n(a)-u_n(z)}{a-z}=-z^n, \text{ when } a=1,$$

so the sum of the series $\sum_{n=1}^{\infty}\frac{u_n(a)-u_n(z)}{a-z}$ is $\frac{-1}{1-z}$, which is infinite for $z=1$.

50. *Connexion of discontinuity with non-uniform convergence.*

We shall now shew that *the sum of a series of continuous functions of z, if it is a uniformly convergent series for values of z within certain limits, cannot be discontinuous for values of z within those limits.*

For let the series be $f(z)=u_1(z)+u_2(z)+\ldots+u_n(z)+\ldots=S_n(z)+R_n(z)$, where R_n is the remainder after n terms.

Since the series is uniformly convergent, we can to any small positive number ϵ find a corresponding integer n independent of z, such that $|R_n(z)|<\frac{\epsilon}{3}$ for all values of z within the area.

Now n and ϵ being thus fixed, we can, on account of the continuity of $S_n(z)$, find a positive number η such that, when $|z-z'|<\eta$, the inequality

$$|S_n(z)-S_n(z')|<\frac{\epsilon}{3}$$

is satisfied.

We have then

$$\begin{aligned}|f(z)-f(z')| &= |\{S_n(z)-S_n(z')\}+R_n(z)-R_n(z')| \\ &< |S_n(z)-S_n(z')|+|R_n(z)|+|R_n(z')| \\ &< \epsilon,\end{aligned}$$

which establishes the result.

Example 1. Shew that at $z=0$ the series

$$u_1(z)+u_2(z)+u_3(z)+\ldots,$$

where $$u_1(z)=z,\quad u_n(z)=z^{\frac{1}{2n-1}}-z^{\frac{1}{2n-3}},$$

and real values of z are concerned, is discontinuous and non-uniformly convergent.

The sum of the first n terms is $z^{\frac{1}{2n-1}}$; as n tends to infinity, this quantity tends to 1, 0, or -1, according as z is positive, zero, or negative. The series is therefore absolutely convergent for all values of z, and has a discontinuity at $z=0$.

The remainder after n terms, when z is small and positive, is $1-z^{\frac{1}{2n-1}}$; however great n may be, by taking $z=e^{-(2n-1)}$ we can cause this remainder to take the value $1-\frac{1}{e}$, which is different from zero. The series is therefore non-uniformly convergent at $z=0$.

Example 2. Shew that at $z=0$ the series

$$\sum_{n=1}^{\infty} \frac{-2z(1+z)^{n-1}}{\{1+(1+z)^{n-1}\}\{1+(1+z)^n\}}$$

is discontinuous and non-uniformly convergent.

The nth term can be written

$$\frac{1-(1+z)^n}{1+(1+z)^n} - \frac{1-(1+z)^{n-1}}{1+(1+z)^{n-1}},$$

so the sum of the first n terms is $\dfrac{1-(1+z)^n}{1+(1+z)^n}$. Thus considering real values of z greater than -1, it is seen that the sum to infinity is 1, 0, or -1, according as z is negative and greater than -2, zero, or positive. There is thus a discontinuity at $z=0$. This discontinuity is explained by the fact that the series is non-uniformly convergent at $z=0$; for the remainder after n terms in the series when z is positive is

$$\frac{-2}{1+(1+z)^n},$$

and however great n may be, by taking $z=\dfrac{1}{n}$ this can be made to take the value $\dfrac{-2}{1+e}$, which is different from zero. The series is therefore non-uniformly convergent at $z=0$.

51. *Distinction between absolute and uniform convergence.*

The *uniform* convergence of a series does not necessitate its *absolute* convergence, nor conversely. Thus the series (§ 49, Ex. 1) $\Sigma \dfrac{z^2}{(1+z^2)^n}$ converges *absolutely*, but (at $z=0$) not *uniformly*: while if we take the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^2+n},$$

its series of moduli is

$$\sum_{n=1}^{\infty} \frac{1}{|n+z^2|},$$

which is divergent, so the series is only *semi-convergent*; but for all real values of z, the terms of the series are alternately positive and negative and numerically decreasing, so the sum of the series lies between the sum of its first n terms and of its first $(n+1)$ terms, and so the remainder after n terms is less than the nth term. Thus we only need take a finite number of terms in order to ensure that for all real values of z the remainder is less than any assigned quantity, i.e. the series is *uniformly* convergent.

Absolutely convergent series behave like series with a finite number of terms in that we can multiply them together and transpose their terms.

Uniformly convergent series behave like series with a finite number of terms in that they are continuous and (as we shall see) can be integrated term by term.

52. *Condition for uniform convergence.*

A sufficient though not *necessary* condition for the uniform convergence of a series may be enunciated as follows :—

If for all values of z within a certain region the moduli of the terms of a series

$$S = u_1(z) + u_2(z) + u_3(z) + \dots$$

are respectively less than the corresponding terms in a convergent series of positive constants

$$T = K_1 + K_2 + K_3 + \dots,$$

then the series S is uniformly convergent in this region. This follows from the fact that, the series T being convergent, it is always possible to choose n so that the remainder after the first n terms of T, and therefore of S, is less than an assigned positive quantity ϵ; and since the value of n thus found is independent of z, the series S is uniformly convergent.

Corollary. The theorem is still true if the moduli of the terms of S, instead of being less than the terms of T, are to them in a variable but finite ratio.

Example. The series

$$\cos z + \frac{1}{2^2}\cos^2 z + \frac{1}{3^2}\cos^3 z + \dots$$

is uniformly convergent for all real values of z, because the moduli of its terms are not greater than the corresponding terms of the convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots,$$

whose terms are positive constants.

53. *Integration of infinite series.*

We shall now shew that if $S(z) = u_1(z) + u_2(z) + \dots$ is a uniformly convergent series of continuous functions of z, for values of z contained within some domain, then the series

$$\int u_1(z)\,dz + \int u_2(z)\,dz + \dots,$$

where all the integrals are taken along with some path C in the domain, is convergent, and has for sum $\int S(z)\,dz$.

For let n be some definite finite number, and write

$$S(z) = u_1(z) + u_2(z) + \dots + u_n(z) + R_n(z),$$

so

$$\int S(z)\,dz = \int u_1(z)\,dz + \dots + \int u_n(z)\,dz + \int R_n(z)\,dz.$$

Now since the series is uniformly convergent, to every positive number ϵ there corresponds a number r independent of z, such that when $n > r$ we have $|R_n(z)| < \epsilon$, for all values of z in the area considered.

Therefore if l be the length of the path of integration, we have (§ 32)

$$\left|\int R_n(z)\,dz\right| < \epsilon l.$$

Therefore the modulus of the difference between $\int S(z)\,dz$ and the sum of the n first terms of the series $\Sigma \int u_n(z)\,dz$ is less than any positive number provided n is large enough. This proves both that the series $\Sigma \int u_n(z)\,dz$ is convergent, and that its sum is $\int S(z)\,dz$.

Example 1. As an example of the necessity of this theorem, consider the series

$$\sum_{n=1}^{\infty} \frac{2z\{n(n+1)\sin^2 z^2 - 1\}\cos z^2}{\{1+n^2\sin^2 z^2\}\{1+(n+1)^2\sin^2 z^2\}}.$$

The nth term is

$$\frac{2zn\cos z^2}{1+n^2\sin^2 z^2} - \frac{2z(n+1)\cos z^2}{1+(n+1)^2\sin^2 z^2},$$

and the sum of n terms is therefore

$$\frac{2z\cos z^2}{1+\sin^2 z^2} - \frac{2z(n+1)\cos z^2}{1+(n+1)^2\sin^2 z^2}.$$

The series is therefore absolutely convergent for all real values of z: but the remainder after n terms is

$$\frac{2z(n+1)\cos z^2}{1+(n+1)^2\sin^2 z^2},$$

and if n be any number however infinitely great, by taking $z = \frac{1}{n+1}$ this has the finite value 2. The series is therefore non-uniformly convergent at $z=0$.

Now the sum to infinity of the series is $\frac{2z\cos z^2}{1+\sin^2 z^2}$, and so the integral from 0 to z of the sum of the series is $\tan^{-1}(\sin z^2)$. On the other hand, the sum of the integrals from 0 to z of the first n terms of the series is

$$\tan^{-1}(\sin z^2) - \tan^{-1}(\overline{n+1}\sin z^2),$$

and for $n=\infty$ this tends to

$$\tan^{-1}(\sin z^2) - \frac{\pi}{2}.$$

Therefore the integral of the sum of the series differs from the sum of the integrals of the terms by $\frac{\pi}{2}$.

Example 2. Discuss the series

$$\sum_{n=1}^{\infty} \frac{2e^n z\{1-n(e-1)+e^{n+1}z^2\}}{n(n+1)(1+e^n z^2)(1+e^{n+1}z^2)}$$

for real values of z.

The nth term of the series may be written

$$\frac{2e^n z}{n(1+e^n z^2)} - \frac{2e^{n+1}z}{(n+1)(1+e^{n+1}z^2)}.$$

The sum of the first n terms is

$$\frac{2ez}{1+ez^2} - \frac{2e^{n+1}z}{(n+1)(1+e^{n+1}z^2)}.$$

The series therefore converges to the value $\frac{2ez}{1+ez^2}$; and since the terms are real and ultimately of the same sign, the convergence is absolute. The integral from 0 to z of the sum of the series is

$$\log(1+ez^2).$$

The sum of the first n terms of the series formed by integrating the terms of the series is

$$\log(1+ez^2) - \frac{1}{n+1}\log(1+e^{n+1}z^2),$$

which for $n=\infty$ tends to

$$\log(1+ez^2) - 1$$

This discrepancy is accounted for by the non-uniform convergence of the series at $z=0$; for the remainder after n terms in the original series is

$$\frac{2e^{n+1}z}{(n+1)(1+e^{n+1}z^2)} \text{ or } \frac{2}{\frac{n+1}{z}e^{-n-1}+(n+1)z},$$

and however great n may be, on taking $z=\frac{1}{n+1}$ this takes the value unity; so the series is non-uniformly convergent at $z=0$.

Example 3. Discuss the series

$$u_1+u_2+u_3+\dots,$$

where

$$u_1 = ze^{-z^2}, \quad u_n = nze^{-nz^2}-(n-1)ze^{-(n-1)z^2},$$

for real values of z.

The sum of the first n terms is nze^{-nz^2}, so the sum to infinity is 0 for all real values of z. Since the terms u_n are real and ultimately all of the same sign, the convergence is absolute.

In the series

$$\int_0^z u_1 dz + \int_0^z u_2 dz + \int_0^z u_3 dz + \dots,$$

the sum of n terms is $\frac{1}{2}(1-e^{-nz^2})$, and this tends to the limit $\frac{1}{2}$ as n tends to infinity; this is not equal to the integral from 0 to z of the sum of the series Σu_n.

The explanation of this discrepancy is to be found in the non-uniformity of the convergence near $z=0$, for the remainder after n terms in the series $u_1+u_2+\dots$ is $-nze^{-nz^2}$; and however great n may be, by taking $z=\frac{1}{n}$ we can cause this to tend to the limit -1, which is different from zero: the series is therefore non-uniformly convergent near $z=0$.

54. *Differentiation of infinite series.*

The converse of the last theorem may be thus stated:

If $S(z) = u_1(z) + u_2(z) + \dots$ is a convergent series of analytic functions of z, which are regular when the variation of z is restricted to be within a certain domain, and if the series $\Sigma(z) = \frac{d}{dz}u_1(z) + \frac{d}{dz}u_2(z) + \dots$ is uniformly convergent within this domain, then this latter series represents $\frac{d}{dz}S(z)$.

For by the preceding result, if a and z are two points within the domain, we have

$$\int_a^z \Sigma(t)\,dt = \int_a^z u_1'(t)\,dt + \int_a^z u_2'(t)\,dt + \dots$$
$$= u_1(z) - u_1(a) + \dots + u_n(z) - u_n(a) + \dots.$$

Since

$$u_1(z) + u_2(z) + \dots \text{ and } u_1(a) + u_2(a) + \dots$$

are each of them convergent series, we can write this

$$\int_a^z \Sigma(t)\,dt = \{u_1(z) + u_2(z) + \dots\} - \{u_1(a) + u_2(a) + \dots\}$$
$$= S(z) - S(a),$$

and hence

$$\Sigma(z) = \frac{d}{dz}S(z).$$

We may note that a derived series may be non-uniformly convergent even when the original series is uniformly convergent: for instance the series

$$\sin z - \tfrac{1}{2}\sin 2z + \tfrac{1}{3}\sin 3z + \dots$$

is non-uniformly convergent at $z = \pi$; although the series from which it can be derived, namely

$$-\cos z + \frac{1}{2^2}\cos 2z - \frac{1}{3^2}\cos 3z + \dots,$$

is uniformly convergent for all real values of z.

55. *Uniform convergence of Power-Series.*

We shall now shew that *a power-series is uniformly convergent at all points within its circle of convergence.*

For let R be a region, forming part of the area of the circle, and let r be a quantity greater than the modulus of every point of R, but less than the radius of convergence. Then if z be any point of R, the moduli of the terms of the series

$$a_0 + a_1 z + a_2 z^2 + \dots$$

are less than the moduli of the corresponding terms of the convergent series

$$a_0 + a_1 r + a_2 r^2 + \dots .$$

But the latter series does not involve z, and so (§ 52) the power-series is uniformly convergent within the region R; as R is arbitrary, the series therefore converges uniformly at all points within the circle of convergence.

It must be observed that nothing is proved regarding points on the circumference; we do not even know that the series is convergent there at all.

Corollary. A power-series is continuous within its circle of convergence: and the series obtained by differentiating and integrating it term by term are equal to the derivate and integral of the function respectively.

Example. As an example of this, consider the series

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots,$$

which is convergent at all points within a circle of radius 1. We can integrate it term by term, so long as the path of integration lies in this circle; the result is

$$\int_0^z \frac{dz}{1+z^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots .$$

Now $\int_0^z \frac{dz}{1+z^2}$ clearly represents that value of $\tan^{-1} z$ which lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$. So the series represents *this* value of $\tan^{-1} z$ and no other.

MISCELLANEOUS EXAMPLES.

1. Shew that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

represents $\frac{1}{(1-z)^2}$ when $|z|<1$ and represents $\frac{1}{z(1-z)^2}$ when $|z|>1$.

Is this fact connected with the theory of uniform convergence?

2. Shew that the series

$$2\sin\frac{1}{3z} + 4\sin\frac{1}{9z} + \dots + 2^n \sin\frac{1}{3^n z} + \dots$$

converges absolutely for all values of z, but does not converge uniformly near $z=0$.

3. If a series $g(z) = \sum_{\nu=0}^{\infty} (c_\nu - c_{\nu+1}) \sin(2\nu+1) z\pi$ (in which c_0 is zero) converges uniformly in an interval, shew that $g(z)\frac{\pi}{\sin \pi z}$ is the derivate of the series

$$f(z) = \sum_{\nu=1}^{\infty} \frac{c_\nu}{\nu} \sin 2\nu z\pi. \qquad \text{(Lerch.)}$$

CHAPTER V.

THE THEORY OF RESIDUES: APPLICATION TO THE EVALUATION OF REAL DEFINITE INTEGRALS.

56. *Residues.*

If a point $z = a$ is a pole of order m for a function $f(z)$, we know by Laurent's theorem that the expansion of the function near $z = a$ is of the form

$$\frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{a_{-1}}{z-a} + \phi(z),$$

where $\phi(z)$ is regular in the vicinity of $z = a$.

The coefficient a_{-1} in this expansion is called the *residue* of the function $f(z)$ relative to the pole a.

Consider now the value of the integral $\int f(z)\,dz$, where the integration is taken round a circle γ, whose centre is the point a and whose radius is a small quantity ρ.

We have $$\int_\gamma f(z)\,dz = \sum_{r=m}^{r=1} a_{-r} \int_\gamma \frac{dz}{(z-a)^r} + \int_\gamma \phi(z)\,dz.$$

Now $\int_\gamma \phi(z)\,dz = 0$, since $\phi(z)$ is a regular function in the interior of the circle γ: and (putting $z - a = \rho e^{i\theta}$) we have

$$\int_\gamma \frac{dz}{(z-a)^r} = \int_0^{2\pi} \frac{\rho e^{i\theta} i\,d\theta}{\rho^r e^{ri\theta}} = \rho^{-r+1} \int_0^{2\pi} e^{\overline{-r+1}\,i\theta}\, i\,d\theta$$

$$= \rho^{-r+1} \left[\frac{e^{\overline{-r+1}\,i\theta}}{-r+1}\right]_0^{2\pi}, \text{ when } r \neq 1$$

$$= 0, \text{ when } r \neq 1.$$

But when $r = 1$ we have

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

Hence finally $$\int_\gamma f(z)\,dz = 2\pi i a_{-1}.$$

Now let C be any contour, containing in the region interior to it a number

of poles $a, b, c, \ldots$ of a function $f(z)$, with residues $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively: and suppose that the function $f(z)$ is regular at all points in the interior of C, except these poles.

Surround the points $a, b, c, \ldots$ by small circles $\alpha, \beta, \gamma, \ldots$: then since the function $f(z)$ is regular in the region bounded by $C, \alpha, \beta, \gamma, \ldots$, its integral taken round the boundary of this region is zero. But this boundary consists of the contour C, described in the positive sense, and the contours $\alpha, \beta, \gamma, \ldots$ described in the negative sense.

Hence $$0 = \int_C f(z)\,dz - \int_\alpha f(z)\,dz - \int_\beta f(z)\,dz \ldots,$$

or $$0 = \int_C f(z)\,dz - 2\pi i a_{-1} - 2\pi i b_{-1} \ldots.$$

Thus we have the *theorem of residues*, namely

$$\int_C f(z)\,dz = 2\pi i \Sigma R,$$

where ΣR denotes the sum of the residues of the function $f(z)$ relative to those of its poles which are situated within the contour C.

This is an extension of the theorem of Chapter III. § 36.

57. *Evaluation of real definite integrals.*

A large number of real definite integrals can be evaluated by the use of contour-integrals and the theorem of residues. The following examples will serve to illustrate the various ways in which these aids to the evaluation may be applied.

Example 1. To find the values of

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\,d\theta \text{ and } \int_0^{2\pi} e^{\cos\theta} \sin(n\theta - \sin\theta)\,d\theta.$$

Denoting these integrals respectively by I and J, we have

$$I - iJ = \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta}\,d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\,d\theta.$$

Write $e^{i\theta} = z$, and let C be a circle of radius unity round the origin in the z-plane. Then as θ assumes the sequence of real values from 0 to 2π, z describes the circle C.

Hence $$I - iJ = \frac{1}{i}\int_C e^z z^{-n-1}\,dz$$

$$= 2\pi \times \text{the residue of } \frac{e^z}{z^{n+1}} \text{ at } z = 0$$

$$= \frac{2\pi}{n!}.$$

Therefore
$$I=\frac{2\pi}{n!},$$
$$J=0.$$

Example 2. The method used in Example 1 can be very generally applied to trigonometrical integrals taken between the limits 0 and 2π. As another example, consider the integral
$$I=\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}\qquad (a>b).$$

Write $e^{i\theta}=z$; and let C be the circle on the z-plane whose centre is at the origin and whose radius is unity.

Then
$$I=\int_C\frac{2dz}{iz\,(2a+bz+bz^{-1})}$$
$$=\frac{2}{i}\int_C\frac{dz}{bz^2+2az+b}$$
$$=4\pi\times\text{sum of residues of }\frac{1}{bz^2+2az+b}\text{ at poles contained within }C.$$

Now
$$\frac{1}{bz^2+2az+b}=\frac{1}{2\sqrt{a^2-b^2}}\left\{\frac{-1}{z+\frac{a}{b}+\frac{\sqrt{a^2-b^2}}{b}}+\frac{1}{z+\frac{a}{b}-\frac{\sqrt{a^2-b^2}}{b}}\right\}.$$

Therefore the two poles are at $z=-\frac{a}{b}-\frac{\sqrt{a^2-b^2}}{b}$ and $z=-\frac{a}{b}+\frac{\sqrt{a^2-b^2}}{b}$, and the residue at the former (which is the only one within C) is $\frac{1}{2\sqrt{a^2-b^2}}$.

Hence
$$I=\frac{2\pi}{\sqrt{a^2-b^2}}.$$

Example 3. Shew that
$$\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}=\frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

Example 4. Find the value of
$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^2+a^2}\,dx.$$

Let C be a contour formed by the real axis together with a semicircle γ, consisting of that half of a circle, whose centre is at the origin and whose radius is very large, which lies above the real axis.

Then $\frac{ze^{mzi}}{z^2+a^2}$ is a function of z which has only one pole in the interior of C, namely at $z=ai$; and so we have $\int_C\frac{ze^{mzi}}{z^2+a^2}dz=2\pi i\times$ residue of $\frac{ze^{mzi}}{z^2+a^2}$ at its pole ai. But writing $z=ai+\zeta$, we have
$$\frac{ze^{mzi}}{z^2+a^2}=\frac{(ai+\zeta)\,e^{-ma+mi\zeta}}{2ai\zeta+\zeta^2}$$
$$=\frac{1}{2}e^{-ma}\frac{1}{\zeta}\left(1+\frac{\zeta}{ai}\right)(1+mi\zeta+\ldots)\left(1+\frac{\zeta}{2ai}\right)^{-1}$$
$$=\frac{e^{-ma}}{2\zeta}+\text{positive powers of }\zeta.$$

Therefore $\dfrac{ze^{mzi}}{z^2+a^2} = \dfrac{e^{-ma}}{2(z-ai)} +$ positive powers of $(z-ai)$.

Thus the residue of $\dfrac{ze^{mzi}}{z^2+a^2}$ at ai is $\dfrac{1}{2}e^{-ma}$.

Therefore $\pi i e^{-ma} = \displaystyle\int_C \frac{ze^{mzi}}{z^2+a^2}dz = \left\{\int_{-\infty}^{\infty} + \int_\gamma\right\} \frac{ze^{mzi}}{z^2+a^2}dz.$

Since $\left|\dfrac{ze^{mzi}}{z^2+a^2}\right|$ is infinitesimal compared with $\dfrac{1}{|z|}$ at points on γ, the integral round γ is infinitesimal compared with $\displaystyle\int_\gamma \left|\frac{dz}{z}\right|$ or 2π, and is therefore zero.

Therefore
$$\pi i e^{-ma} = \int_{-\infty}^{\infty} \frac{ze^{mzi}}{z^2+a^2}dz.$$

Equating imaginary parts, we have
$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2}dx = \pi e^{-ma}.$$

Example 5. To find the value of
$$\int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{x\,dx}{x^2+r^2}.$$

We have
$$\int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{x\,dx}{x^2+r^2} = \int_{-\infty}^{\infty} \frac{1}{2i} e^{ae^{bxi}} \frac{x\,dx}{x^2+r^2}.$$

Take a contour C composed as in Example 4 of an infinite semicircle γ and the real axis.

Then $\displaystyle\int_C \frac{1}{2i} e^{ae^{bzi}} \frac{z}{z^2+r^2}dz = 2\pi i \times$ residue of $\dfrac{1}{2i} e^{ae^{bzi}} \dfrac{z}{z^2+r^2}$ at its poles inside C.

But $\dfrac{1}{2i} e^{ae^{bzi}} \dfrac{z}{z^2+r^2}$ has only one pole in the interior of C, namely at the point $z=ri$.

Now if $z=ri+\zeta$, we have
$$\frac{1}{2i} e^{ae^{bzi}} \frac{z}{z^2+r^2} = \frac{1}{2i} e^{ae^{-br-b\zeta i}} \frac{ri+\zeta}{2ri\zeta+\zeta^2} = \frac{1}{4i\zeta} e^{ae^{-br}} + \text{positive powers of } \zeta.$$

Therefore the residue is
$$\frac{1}{4i} e^{ae^{-br}}.$$

Thus
$$\frac{\pi}{2} e^{ae^{-br}} = \int_\gamma \frac{1}{2i} e^{ae^{bzi}} \frac{z}{z^2+r^2}dz + \int_{-\infty}^{\infty} \frac{1}{2i} e^{ae^{bxi}} \frac{x}{x^2+r^2}dx.$$

But at points on γ, $e^{bxi}=0$, so $e^{ae^{bxi}}=1$, and so
$$\int_\gamma \frac{1}{2i} e^{ae^{bxi}} \frac{x\,dx}{x^2+r^2} = \frac{1}{2i}\int_\gamma \frac{dx}{x} = \frac{\pi}{2}.$$

Therefore
$$\frac{\pi}{2} e^{ae^{-br}} = \frac{\pi}{2} + \int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{x\,dx}{x^2+r^2},$$

or
$$\int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{x\,dx}{x^2+r^2} = \frac{\pi}{2}\left(e^{ae^{-br}} - 1\right).$$

We may note that in the above $\displaystyle\int_{-\infty}^{\infty}$ stands for the limit of $\displaystyle\int_{-k}^{k}$ where k is infinitely great, and is not equal to the limit of $\displaystyle\int_{-l}^{k}$ where k and l are different.

Example 6. Prove by integrating

$$\int \frac{e^{iz}\,dz}{a^2+z^2}$$

round the contour used in Examples 4 and 5, that

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2+x^2} = \frac{\pi}{2}e^{-a}, \qquad \int_{-\infty}^{\infty} \frac{\sin x}{a^2+x^2} = 0.$$

Example 7. Find the value of

$$\int_0^\infty \frac{\sin mx}{x(x^2+a^2)^2}\,dx.$$

Consider a contour C, formed of 1° a semicircle Γ whose centre is at the origin and whose radius is very large, 2° a semicircle γ whose centre is also at the origin and whose radius is very small, and 3° the portions of the real axis intercepted between these circles. The semicircles are to be drawn in the upper half of the z-plane, i.e. the half above the real axis. Then $\displaystyle\int_C \frac{e^{miz}\,dz}{z(z^2+a^2)^2} = 2\pi i \times$ the residue of $\dfrac{e^{miz}}{z(z^2+a^2)}$ at the singularity $z=ai$.

But if we write $z = ai+\zeta$, where ζ is small, we have

$$\frac{e^{miz}}{z(z^2+a^2)^2} = \frac{e^{-ma+mi\zeta}}{(ai+\zeta)(2ai\zeta+\zeta^2)^2} = \frac{ie^{-ma}}{2^2a^3\zeta^2}(1+mi\zeta+\ldots)\left(1+\frac{\zeta}{ai}\right)^{-1}\left(1+\frac{\zeta}{2ai}+\ldots\right)^{-2}$$

$$= \frac{ie^{-ma}}{4a^3\zeta^2}\left\{1+\left(m+\frac{2}{a}\right)i\zeta+\ldots\right\} = \frac{ie^{-ma}}{4a^3\zeta^2} - \frac{e^{ma}}{4a^3}\left(m+\frac{2}{a}\right)\frac{1}{\zeta}+\ldots.$$

Thus the residue at ai is $-\dfrac{e^{-ma}}{4a^3}\left(m+\dfrac{2}{a}\right)$.

Therefore $$-\frac{i\pi e^{-ma}}{2a^3}\left(m+\frac{2}{a}\right) = \int_C \frac{e^{miz}\,dz}{z(z^2+a^2)^2} = \left\{\int_{-\infty}^{\infty} + \int_\Gamma - \int_\gamma\right\}\frac{e^{miz}\,dz}{z(z^2+a^2)^2}.$$

Now $\dfrac{e^{miz}}{(z^2+a^2)^2}$ is infinitely small at points on Γ, so the integral taken round Γ vanishes.

Also $$\int_\gamma \frac{e^{miz}\,dz}{z(z^2+a^2)^2} = \int \frac{dz}{z}\left\{\frac{1}{a^4} + \text{powers of } z\right\} = \frac{1}{a^4}\int_\gamma \frac{dz}{z} = \frac{\pi i}{a^4}.$$

Therefore $$\int_{-\infty}^{\infty} \frac{e^{miz}\,dz}{z(z^2+a^2)^2} = \frac{\pi i}{a^4} - \frac{i\pi e^{-ma}}{2a^3}\left(m+\frac{2}{a}\right).$$

In this, $\displaystyle\int_{-\infty}^{\infty}$ means $\displaystyle\int_\epsilon^\infty + \int_{-\infty}^{-\epsilon}$, where the two ϵ's are the same: but in the final result we can put $\epsilon=0$, since the final integrand is finite at the origin.

Equating the imaginary parts on both sides of this equation, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin mx\,dx}{x(x^2+a^2)^2} = \frac{\pi}{a^4} - \frac{\pi e^{-ma}}{2a^3}\left(m+\frac{2}{a}\right),$$

and so $$\int_0^\infty \frac{\sin mx\,dx}{x(x^2+a^2)^2} = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3}\left(m+\frac{2}{a}\right).$$

Example 8. Find the value of

$$\int_{-\infty}^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\,dx.$$

Take the contour C formed as in Example 7 by an infinite semicircle Γ, a small semi-

circle γ round the origin, and the parts of the real axis intercepted between them. Within this contour the function $\dfrac{e^{2aiz}}{z^2}$ has no singularities.

Therefore $$0=\int_C \frac{e^{2aiz}}{z^2}\,dz=\int_\Gamma \frac{e^{2aiz}}{z^2}\,dz-\int_\gamma \frac{e^{2aiz}}{z^2}\,dz+\int_{-\infty}^{\infty}\frac{e^{2aiz}}{z^2}\,dz.$$

In this equation $\int_{-\infty}^{\infty}$ must be regarded as an abbreviation for $\int_\epsilon^\infty+\int_{-\infty}^{-\epsilon}$ where ϵ is the radius of γ.

Now at points on Γ, $\dfrac{e^{2aiz}}{z^2}$ is zero compared with $\dfrac{1}{z}$, so the integral round Γ is zero.

Also $$\int_\gamma \frac{e^{2aiz}}{z^2}\,dz=\text{one-half of } 2\pi i\times\text{the residue of } \frac{e^{2aiz}}{z^2} \text{ at the origin}$$
$$=\pi i\times\text{the residue of } \frac{1+2aiz+\dots}{z^2}$$
$$=-2\pi a.$$

Therefore $$\int_{-\infty}^{\infty}\frac{e^{2aiz}}{z^2}\,dz=-2\pi a,$$

and so $$\int_{-\infty}^{\infty}\frac{e^{2aiz}-e^{2biz}}{z^2}\,dz=2\pi(b-a).$$

Taking the real part of this we have
$$\int_{-\infty}^{\infty}\frac{\cos 2ax-\cos 2bx}{x^2}\,dx=2\pi(b-a),$$
and since $\dfrac{\cos 2ax-\cos 2bx}{x^2}$ is finite when $x=0$, we need no longer restrict $\int_{-\infty}^{\infty}$ to mean $\int_\epsilon^\infty+\int_{-\infty}^{-\epsilon}$.

Example 9. Find the value of
$$\int_0^\infty x^{a-1}\sin\left(\frac{a\pi}{2}-bx\right)\frac{r\,dx}{x^2+r^2}\qquad(a>0).$$

We have $$\int_0^\infty x^{a-1}\sin\left(\frac{a\pi}{2}-bx\right)\frac{r\,dx}{x^2+r^2}$$
$$=\frac{1}{2}\int_0^\infty \frac{x^{a-1}}{i}\left\{e^{\frac{ai\pi}{2}-bix}-e^{-\frac{ai\pi}{2}+bix}\right\}\frac{r\,dx}{x^2+r^2}$$
$$=\frac{1}{2}i^{a-1}\int_0^\infty\{x^{a-1}e^{-ibx}+(-x)^{a-1}e^{ibx}\}\frac{r\,dx}{x^2+r^2}$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}(-xi)^{a-1}e^{ibx}\frac{r\,dx}{x^2+r^2}.$$

Consider a contour C, formed as in Examples 7 and 8 by an infinite semicircle Γ, a small semicircle γ round the origin, and the parts of the real axis intercepted between them.

Then $\dfrac{1}{2}\displaystyle\int_C(-zi)^{a-1}e^{ibz}\frac{r\,dz}{z^2+r^2}=2\pi i\times$ the residue of $\dfrac{1}{2}(-zi)^{a-1}e^{ibz}\dfrac{r}{z^2+r^2}$ at its singularity $z=ir$.

Putting $z=ri+\zeta$ and neglecting powers of ζ, we see that the expansion of

$$\frac{1}{2}(-zi)^{a-1}e^{ibz}\frac{r}{z^2+r^2}$$

begins with a term
$$-\frac{i}{4}\frac{r^{a-1}e^{-br}}{\zeta},$$

so the required residue is
$$-\frac{i}{4}r^{a-1}e^{-br}.$$

Therefore
$$\frac{\pi}{2}r^{a-1}e^{-br}=\frac{1}{2}\int_C(-zi)^{a-1}e^{ibz}\frac{r\,dz}{z^2+r^2}$$
$$=\frac{1}{2}\left\{\int_{-\infty}^{\infty}+\int_\Gamma-\int_\gamma\right\}(-zi)^{a-1}e^{ibz}\frac{r\,dz}{z^2+r^2}.$$

At points on Γ the integrand is infinitesimal compared with $\frac{1}{z}$, and so the integral round Γ is zero.

At points on γ the integrand is approximately $\frac{(-i)^{a-1}}{2r}z^{a-1}$, and so if $a>0$ the integral round γ is zero.

Therefore
$$\int_0^\infty x^{a-1}\sin\left(\frac{a\pi}{2}-bx\right)\frac{r\,dx}{x^2+r^2}=\frac{1}{2}\int_{-\infty}^{\infty}(-xi)^{a-1}e^{ibx}\frac{r\,dx}{x^2+r^2}=\frac{\pi}{2}r^{a-1}e^{-br}.$$

Example 10. Find the value of $\int_0^\infty e^{a\cos bx}\sin(a\sin bx)\frac{dx}{x}$.

We have
$$\int_0^\infty e^{a\cos bx}\sin(a\sin bx)\frac{dx}{x}=\frac{1}{2i}\int_{-\infty}^{\infty}e^{ae^{bxi}}\frac{dx}{x},$$

where in the latter integral $\int_{-\infty}^{\infty}$ must be regarded as an abbreviation for $\int_\epsilon^\infty+\int_{-\infty}^{-\epsilon}$ where ϵ is a small quantity.

Take a contour C, consisting as in Examples 7, 8, 9, of an infinite semicircle Γ, a small semicircle γ of radius ϵ round the origin, and the parts of the real axis intercepted between them.

Then
$$0=\int_C e^{ae^{bxi}}\frac{dx}{x}=\int_\Gamma e^{ae^{bxi}}\frac{dx}{x}-\int_\gamma e^{ae^{bxi}}\frac{dx}{x}+\int_{-\infty}^{\infty}e^{ae^{bxi}}\frac{dx}{x}.$$

At points on Γ, we have $e^{bxi}=0$, $e^{ae^{bxi}}=1$, and so

$$\int_\Gamma e^{ae^{bxi}}\frac{dx}{x}=\int_\Gamma\frac{dx}{x}=\pi i.$$

At points on γ, $e^{bxi}=1$, so

$$\int_\gamma e^{ae^{bxi}}\frac{dx}{x}=e^a\int_\gamma\frac{dx}{x}=\pi ie^a.$$

Therefore
$$\int_{-\infty}^{\infty}e^{ae^{bxi}}\frac{dx}{x}=\pi i(e^a-1),$$

and so
$$\int_0^\infty e^{a\cos bx}\sin(a\sin bx)\frac{dx}{x}=\frac{\pi}{2}(e^a-1).$$

Example 11. By integrating $\int\frac{e^{iz}dz}{z}$ round the same contour as that used in Examples 7, 8, 9, 10, shew that $\int_0^\infty\frac{\sin x}{x}dx=\frac{\pi}{2}$.

Example 12. To find the value of

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx, \text{ and } \int_0^\infty \frac{x^{a-1}}{1-x}\,dx \qquad (0<a<1).$$

Write
$$I=\int_0^\infty \frac{x^{a-1}}{1+x}\,dx, \text{ and } K=\int_0^\infty \frac{x^{a-1}}{1-x}\,dx.$$

As will be seen from the working below, the integral K has a meaning only when $\int_0^\infty$ is understood to mean $\int_{1+r'}^\infty + \int_0^{1-r'}$, where r' is a small positive quantity.

Consider a contour C formed of (a) that half Γ of a circle, whose centre is at the origin and whose radius is a large quantity R, which is above the real axis, (b) that half γ of a circle whose centre is at the origin and whose radius is a small quantity r, which is above the real axis, (c) that half γ' of a circle, whose centre is at the point (-1) and whose radius is a small quantity r', which is above the real axis, (d) the parts of the real axis intercepted between these semicircles.

Consider $\int_C \frac{z^{a-1}dz}{1+z}$, where the many-valued function is supposed to have that one of its determinations which is real and positive when z is real and positive. The integrand is regular in the interior of the contour C, and so

$$0=\int_C \frac{z^{a-1}dz}{1+z},$$

or
$$\left(\int_{-\infty}^{-1-r'} + \int_{-1+r'}^{+\infty}\right)\frac{x^{a-1}dx}{1+x} = \left(\int_\gamma + \int_{\gamma'} - \int_\Gamma\right)\frac{z^{a-1}dz}{1+z}.$$

Now on γ the integrand is sensibly equal to z^{a-1}, and so the integral to $\left[\frac{z^a}{a}\right]_{-r}^{r}$, which is infinitesimal, since $a>0$.

On γ', the integrand is sensibly equal to $\frac{(-1)^{a-1}}{1+z}$; putting $1+z=r'e^{i\theta}$, the integral along γ' is $\int_0^\pi i\,d\theta$, or $i\pi(-1)^{a-1}$.

On Γ, the integrand is sensibly equal to $\frac{1}{z^{2-a}}$, the modulus of which is infinitesimal compared with $\frac{1}{|z|}$; so the integral along Γ is zero.

Therefore
$$(-1)^{a-1}\pi i=\int_0^\infty \frac{x^{a-1}dx}{1+x} + \left(\int_{-\infty}^{-1-r'} + \int_{-1+r'}^{0}\right)\frac{x^{a-1}dx}{1+x} = I+(-1)^{a-1}K.$$

Thus
$$\pi i=(-1)^{1-a}I+K=-I(\cos a\pi - i\sin a\pi)+K.$$

Therefore equating real and imaginary parts, we have

$$I=\frac{\pi}{\sin a\pi},$$

$$K=\pi\cot a\pi.$$

Example 13. By using the result

$$\int_0^\infty \frac{x^{a-1}dx}{1+x}=\frac{\pi}{\sin a\pi},$$

shew that
$$\frac{e^{2\pi i v x}}{1-e^{2\pi i x}}=\frac{1}{2\pi i}\cdot \underset{n=\infty}{\text{Limit}} \sum_{k=-n}^{n} \frac{e^{2k v\pi i}}{k-x}. \qquad \text{(Kronecker.)}$$

58. *Evaluation of the definite integral of a rational function.*

The principles which have been applied in the preceding paragraph can also be used to evaluate an integral of the form

$$\int_{-\infty}^{\infty} f(x)\, dx,$$

where $f(x)$ is a rational function of x, in the cases when this integral has a meaning.

For suppose that $f(x)$ is brought to the form of a quotient $\dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials in x. In order that the integral may have a meaning unconditionally, it is necessary that the degree of $g(x)$ should be at least two units lower than that of $h(x)$, and that the equation $h(x)=0$ should have no real roots.

Consider now a contour C, formed of the real axis together with a semicircle Γ of large radius, whose centre is at the origin, and which lies in the upper half of the z-plane.

We have $\int_C f(z)\, dz = 2\pi i \times$ sum of residues of $f(z)$ at the poles of $f(z)$ contained within C.

Now $\int_C = \int_{-\infty}^{\infty} + \int_\Gamma$: and since $f(z)$ has a zero of at least order 2 at $z=\infty$, it follows that $\int_\Gamma$ is zero.

Hence $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \times$ sum of residues of $f(x)$ at those of its poles which are contained in the upper half of the z-plane.

If the degree of $g(x)$ is lower than that of $h(x)$ by only one degree, or if $h(x)$ has real non-repeated roots, the integral will still have a meaning provided we make certain restrictions, i.e. that $\int_{-\infty}^{\infty}$ shall be understood to mean the limit, when k tends to ∞ and ϵ to zero, of $\int_{-k}^{c-\epsilon} + \int_{c+\epsilon}^{k}$, where c is a typical root of the equation $h(x)=0$.

Example 1. The function $\dfrac{1}{(z^2+1)^3}$ has a single pole in the upper half of the z-plane, namely at $z=i$, and the residue there is $-\dfrac{3i}{16}$; we have therefore

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}.$$

Example 2. Shew that $$\int_{-\infty}^{\infty} \frac{x^4\, dx}{(a+bx^2)^4} = \frac{\pi}{16 a^{\frac{3}{2}} b^{\frac{5}{2}}}.$$

59. *Cauchy's integral.*

We shall next discuss a class of contour-integrals which are very frequently found useful in analytical investigations.

Let C be a contour in the z-plane, and let $f(z)$ be a function regular everywhere in the interior of C. Let $\phi(z)$ be another function, which in the interior of C has no singularities except poles; let the zeros of $\phi(z)$ in the interior of C be $a_1, a_2, \ldots$, and let their degrees of multiplicity be $r_1, r_2, \ldots$; and let its poles in the interior of C be $b_1, b_2, \ldots$, and let their degrees of multiplicity be $s_1, s_2, \ldots$.

Then by the fundamental theorem on residues, we have

$$\frac{1}{2\pi i}\int_C f(z)\frac{\phi'(z)}{\phi(z)}dz = \text{sum of residues of } \frac{f(z)\phi'(z)}{\phi(z)} \text{ in the interior of } C.$$

Now $\frac{f(z)\phi'(z)}{\phi(z)}$ can have singularities only at the poles and zeros of $\phi(z)$. At one of the zeros, say a_1, we have

$$\phi(z) = A(z-a_1)^{r_1} + B(z-a_1)^{r_1+1} + \ldots.$$

Therefore $\phi'(z) = Ar_1(z-a_1)^{r_1-1} + B(r_1+1)(z-a_1)^{r_1} + \ldots,$

and $$f(z) = f(a_1) + (z-a_1)f'(a_1) + \ldots.$$

Therefore $\frac{f(z)\phi'(z)}{\phi(z)} = \frac{r_1 f(a_1)}{z-a_1}$ + a constant + positive powers of $(z-a_1)$.

Thus the residue of $\frac{f(z)\phi'(z)}{\phi(z)}$, at the point $z=a_1$, is $r_1 f(a_1)$.

Similarly the residue at $z=b_1$ is $-s_1 f(b_1)$; for near $z=b_1$, we have

$$\phi(z) = C(z-b_1)^{-s_1} + D(z-b_1)^{-s_1+1} + \ldots,$$

and $$f(z) = f(b_1) + (z-b_1)f'(b_1) + \ldots,$$

so $\frac{f(z)\phi'(z)}{\phi(z)} = \frac{-s_1 f(b_1)}{z-b_1}$ + a constant + positive powers of $z-b_1$.

Hence $$\frac{1}{2\pi i}\int_C f(z)\frac{\phi'(z)}{\phi(z)}dz = \Sigma r_1 f(a_1) - \Sigma s_1 f(b_1),$$

the summations being extended over all the zeros and poles of $\phi(z)$.

60. *The number of roots of an equation contained within a contour.*

The result of the preceding paragraph can be at once applied to find the number of roots of an equation $\phi(z)=0$ contained within a contour C.

For on putting $f(z)=1$ in the preceding result, we obtain the result that $\frac{1}{2\pi i}\int_C \frac{\phi'(z)}{\phi(z)}dz$ is equal to the excess of the number of zeros over the number

of poles of $\phi(z)$ contained in the interior of C, each pole and zero being reckoned according to its degree of multiplicity.

Example 1. Shew that a polynomial $\phi(z)$ of degree m has m roots.

Let $$\phi(z)=a_0z^m+a_1z^{m-1}+\ldots+a_m.$$

Then $$\frac{\phi'(z)}{\phi(z)}=\frac{ma_0z^{m-1}+\ldots+a_{m-1}}{a_0z^m+\ldots+a_m}.$$

For large values of z, this can be expanded in the form
$$\frac{\phi'(z)}{\phi(z)}=\frac{m}{z}+\frac{A}{z^2}+\ldots.$$

Thus if C be a large circle whose centre is at the origin, we have
$$\frac{1}{2\pi i}\int_C\frac{\phi'(z)}{\phi(z)}dz=\frac{m}{2\pi i}\int_C\frac{dz}{z}=m.$$

Hence as $\phi(z)$ has no poles in the interior of C, we have
$$\text{number of zeros of }\phi(z)=\frac{1}{2\pi i}\int_C\frac{\phi'(z)}{\phi(z)}dz=m.$$

Example 2. If at all points of a contour C the inequality
$$|a_kz^k|>|a_0+a_1z+\ldots+a_{k-1}z^{k-1}+a_{k+1}z^{k+1}+\ldots+a_mz^m$$
is satisfied, then the contour contains k roots of the equation
$$a_mz^m+a_{m-1}z^{m-1}+\ldots+a_1z+a_0=0.$$

For write $$f(z)=a_mz^m+a_{m-1}z^{m-1}+\ldots+a_1z+a_0.$$

Then $$f(z)=a_kz^k\left(1+\frac{a_mz^m+\ldots+a_{k+1}z^{k+1}+a_{k-1}z^{k-1}+\ldots+a_0}{a_kz^k}\right)$$
$$=a_kz^k(1+U)\text{ say, where }|U|<1\text{ on the contour.}$$

Therefore the number of roots of $f(z)$ contained in C
$$=\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}dz$$
$$=\frac{1}{2\pi i}\int_C\left(\frac{k}{z}+\frac{1}{1+U}\frac{dU}{dz}\right)dz.$$

But $\int_C\frac{dz}{z}=2\pi i$; and since $|U|<1$ we can expand $(1+U)^{-1}$ in the form
$$1-U+U^2-U^3+\ldots,$$
so $$\int_C\frac{1}{1+U}\frac{dU}{dz}dz=\left[U-\tfrac{1}{2}U^2+\tfrac{1}{3}U^3-\ldots\right]_C=0.$$

Therefore the number of roots contained in C is equal to k.

61. *Connexion between the zeros of a function and the zeros of its derivate.*

Macdonald* has shewn that *if $f(z)$ be a regular function of z in the interior of a contour C, defined by an equation $|f(z)|=M$ where M is a constant, then the number of zeros of $f(z)$ in this region exceeds the number of zeros of the derived function $f'(z)$ in the same region by unity.*

* *Proc. Lond. Math. Soc.* XXIX. (1898).

For since $f(z)$ has no essential singularity in the region, the number N of its zeros in the region is finite. Now if m be a small number, the part of the locus $|f(z)|=m$ in the interior of the contour C consists of N closed curves surrounding the N zeros of $f(z)$. As m increases, these ovals increase, until two of them coalesce, the point at which they coalesce being a node on the curve corresponding to that particular value of m. When m has increased to its final value M, the N closed curves have coalesced into one closed curve, and therefore $N-1$ nodes have been passed through. Each of these nodes is a zero of $f'(z)$; for if $f(z)=\phi+i\psi$, where ϕ and ψ are functions of x and y with real coefficients, then $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ vanish at a node on the curve $\phi^2+\psi^2=\text{constant}$; that is, $f'(z)$ vanishes. Moreover, two ovals cannot coalesce at more than one point, as $f(z)$ is single-valued.

Hence the number of zeros of $f'(z)$ inside the contour is $(N-1)$.

The proof assumes the zeros of $f(z)$ in the interior of C to be all simple: the case where $f(z)$ has multiple zeros can be at once reduced to this, by dividing out the factor common to $f(z)$ and $f'(z)$. If $f'(z)$ has two zeros equal, two of the double points coalesce, that is, three ovals coalesce at the same point.

Similarly it can be shewn that the number of zeros of $f'(z)$ in the region between the contours $|f(z)|=m_1$ and $|f(z)|=m_2$ is equal to the number of zeros of $f(z)$ in the same region, if $f(z)$ is regular in the region.

Example 1. Deduce from Macdonald's result the theorem that a polynomial of degree n has n zeros.

Example 2. Deduce from Macdonald's result that if a function $f(z)$, regular for real finite values of z, has all its coefficients real, and all its zeros real and different, then between two consecutive zeros of $f(z)$ there is one zero and one only of $f'(z)$.

Miscellaneous Examples.

1. A function $\phi(z)$ is zero for $z=0$ and regular when $|z|\leqslant 1$. If $f(x, y)$ is the coefficient of i in $\phi(x+yi)$, prove that

$$\int_0^{2\pi}\frac{x\sin\theta}{1-2x\cos\theta+x^2}f(\cos\theta,\ \sin\theta)\,d\theta=\pi\phi(x).$$

(Trinity College Examination, 1898.)

2. Shew that $$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx=\frac{1}{4}\frac{e^a+1}{e^a-1}-\frac{1}{2a}.$$ (Legendre.)

3. By integrating $\int e^{-z^2}dz$ round the perimeter of a rectangle of which one side is the real axis and another side is parallel to the real axis and at a distance a from it, shew that

$$\int_{-\infty}^{\infty} e^{-t^2}\cos 2at\,dt=\sqrt{\pi}\,e^{-a^2},$$

and

$$\int_{-\infty}^{\infty} e^{-t^2}\sin 2at\,dt=0.$$

4. Shew that $$\int_0^{\frac{\pi}{2}}\frac{1-r\cos 2\theta}{1-2r\cos 2\theta+r^2}\log\sin\theta\,d\theta=\frac{\pi}{4}\log\frac{1-r}{4}.$$

5. Shew that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1-2a\cos x+a^2}\, x\,dx = \frac{\pi}{4a}\log(1+a) \text{ if } -1<a<1$$

and

$$= \frac{\pi}{4a}\log\left(1+\frac{1}{a}\right) \text{ if } a^2>1.$$

(Cauchy.)

6. Shew that

$$\int_0^\infty \frac{\sin\phi_1 x}{x}\frac{\sin\phi_2 x}{x}\cdots\frac{\sin\phi_n x}{x}\cos a_1 x\ldots\cos a_m x\,\frac{\sin ax}{x}\,dx = \frac{\pi}{2}\phi_1\phi_2\ldots\phi_n,$$

if a be different from zero and

$$a > |\phi_1| + |\phi_2| + \ldots + |\phi_n| + |a_1| + \ldots + |a_m|.$$

(Störmer.)

7. If a point z describes a circle C of centre a, any one-valued function $u=f(z)$ will describe a closed curve γ in the u-plane. Shew that if to each element of γ be attributed a mass proportional to the corresponding element of C, the centre of gravity of γ is the point r, where r is the sum of the residues of $\dfrac{f(z)}{z-a}$ at poles in the interior of C.

(Amigues.)

8. Shew that

$$\int_{-\infty}^\infty \frac{dx}{(x^2+b^2)(x^2+a^2)^2} = \frac{\pi(2a+b)}{2a^2b(a+b)^2}.$$

9. Shew that

$$\int_{-\infty}^\infty \frac{dx}{(a+bx^2)^n} = \frac{\pi}{2^{n-1}b^{\frac{1}{2}}}\,\frac{1.3\ldots(2n-3)}{1.2\ldots(n-1)}\,\frac{1}{a^{n-\frac{1}{2}}}.$$

10. If $F_n(x)=(1-x)(1-x^2)\ldots(1-x^{n-1})\ldots(1-x^2)(1-x^4)\ldots(1-x^{2n-2})\ldots$
$\ldots(1-x^{n-1})(1-x^{2n-2})\ldots(1-x^{(n-1)^2})$,

shew that the series

$$f(x) = -\sum_{n=2}^{n=\infty}\frac{F_n\left(\dfrac{x}{n}\right)}{\left(\dfrac{x^n}{n^n}-1\right)n^{n-1}}$$

converges when x is not a root of one of the equations

$$\left(\frac{x}{n}\right)^n - 1 = 0;$$

and that the sum of the residues of $f(x)$ contained in the ring-shaped space included between two circles whose centres are at the origin, one having a small radius and the other having a radius between n and $n+1$, is equal to the number of prime numbers less than $n+1$.

(Laurent.)

CHAPTER VI.

The Expansion of Functions in Infinite Series.

62. *Darboux's formula.*

Darboux has given* a formula from which a large number of expansions in infinite series can be derived.

Let $f(z)$ be an analytic function of z, regular at all points z within a circle of centre a and radius r; and let z be a point within this circle. Let $\phi(z)$ be any polynomial in z, of degree n. Then if R_n denotes the expression

$$(-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)} \{a + t(z-a)\} \, dt,$$

where the integration is taken along the real axis of t, we have on integration by parts

$$R_n = \Big[(-1)^n (z-a)^n \phi(t) f^{(n)} \{a + t(z-a)\}\Big]_0^1 \\ + (-1)^{n-1} (z-a)^n \int_0^1 \phi'(t) f^{(n)} \{a + t(z-a)\} \, dt,$$

or $$R_n = (-1)^n (z-a)^n \{\phi(1) f^{(n)}(z) - \phi(0) f^{(n)}(a)\} \\ + (-1)^{n-1} (z-a)^n \int_0^1 \phi'(t) f^{(n)} \{a + t(z-a)\} \, dt.$$

Integrating the last integral by parts in the same way, we obtain

$$R_n = (-1)^n (z-a)^n \{\phi(1) f^{(n)}(z) - \phi(0) f^{(n)}(a)\} \\ + (-1)^{n-1} (z-a)^{n-1} \{\phi'(1) f^{(n-1)}(z) - \phi'(0) f^{(n-1)}(a)\} + \dots \\ - (z-a) \{\phi^{(n-1)}(1) f'(z) - \phi^{(n-1)}(0) f'(a)\} \\ + (z-a) \int_0^1 \phi^{(n)}(t) f' \{a + t(z-a)\} \, dt.$$

Now $\phi^{(n)}(t)$ is a constant independent of t, since $\phi(t)$ is a polynomial of order n; and hence

$$(z-a) \int_0^1 \phi^{(n)}(t) f' \{a + t(z-a)\} \, dt = \phi^n(0) \{f(z) - f(a)\}.$$

* *Liouville's Journal* (3), II. (1876), p. 271.

Thus finally we have Darboux's formula

$$\phi^{(n)}(0)\{f(z)-f(a)\}=(z-a)\{\phi^{(n-1)}(1)f'(z)-\phi^{(n-1)}(0)f'(a)\}\ldots$$
$$+(-1)^n(z-a)^{n+1}\int_0^1 \phi(t) f^{(n+1)}\{a+t(z-a)\}\,dt.$$

Taylor's expansion may be derived from this formula by putting $\phi(t)=(t-1)^n$, and then making n tend to infinity: other new expansions may be obtained by substituting special polynomials of degree n for $\phi(t)$, and in the resulting formula making n tend to infinity: in each case it must of course be shewn that R_n tends to zero as n tends to infinity.

Example. By substituting $2n$ for n in Darboux's formula, and taking $\phi(t)=t^n(t-1)^n$, obtain the expansion

$$f(z)-f(a)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(z-a)^n}{2^n n!}\{f^{(n)}(z)+(-1)^{(n-1)}f^{(n)}(a)\},$$

and find the expression for the remainder after n terms in this series.

63. *The Bernoullian numbers and the Bernoullian polynomials.*

If the constants which occur in the expansion of $\frac{z}{2}\cot\frac{z}{2}$ in ascending powers of z be denoted by $B_1, B_2, B_3, \ldots$, so that

$$\frac{z}{2}\cot\frac{z}{2}=1-B_1\frac{z^2}{2!}-B_2\frac{z^4}{4!}-B_3\frac{z^6}{6!}\ldots,$$

then B_n is called the nth *Bernoullian number.* It is found that

$$B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42},\ \ldots.$$

The Bernoullian numbers can be expressed as definite integrals in the following way.

We have
$$\int_0^\infty \frac{\sin px\,dx}{e^{\pi x}-1}=\sum_{n=1}^{\infty}\int_0^\infty e^{-n\pi x}\sin px\,dx$$
$$=\sum_{n=1}^{\infty}\frac{p}{n^2\pi^2+p^2}$$
$$=-\frac{1}{2p}-\frac{i}{2}\cot ip$$
$$=-\frac{1}{2p}+\frac{1}{2p}\left\{1+B_1\frac{(2p)^2}{2!}-B_2\frac{(2p)^4}{4!}\ldots\right\}.$$

Equating coefficients of p^{2n-1} on the two sides of this equation, and writing $x=2t$, we obtain

$$B_n=4n\int_0^\infty \frac{t^{2n-1}\,dt}{e^{2\pi t}-1}.$$

A proof of this result, depending on contour integration, is given by Carda, *Monatshefte für Math. und Phys.* v. (1894), pp. 321–4.

Example. Shew that

$$B_n = \frac{2n}{\pi^{2n}(2^{2n}-1)} \int_0^\infty \frac{x^{2n-1}dx}{\sinh x}.$$

The *Bernoullian polynomial of order n* is defined to be the coefficient of $\frac{t^n}{n!}$ in the expansion of $t\frac{e^{zt}-1}{e^t-1}$ in ascending powers of t. It is denoted by $\phi_n(z)$, so that

$$t\frac{e^{zt}-1}{e^t-1} = \sum_{n=1}^\infty \frac{\phi_n(z)\,t^n}{n!} \quad \text{.............................(1)}.$$

This function possesses several important properties. Writing $(z+1)$ for z in the preceding equation and taking the difference of the two results, we have

$$te^{zt} = \sum_{n=1}^\infty \{\phi_n(z+1) - \phi_n(z)\}\frac{t^n}{n!}.$$

On equating coefficients of t^n on both sides of this equation we obtain

$$nz^{n-1} = \phi_n(z+1) - \phi_n(z),$$

which is a difference-equation satisfied by the function $\phi_n(z)$.

The explicit expression of the Bernoullian polynomials can be obtained as follows. We have

$$e^{zt} - 1 = zt + \frac{z^2t^2}{2!} + \frac{z^3t^3}{3!} + \dots,$$

and

$$\frac{t}{e^t-1} = \frac{t}{2}\frac{e^t+1}{e^t-1} - \frac{t}{2}$$

$$= \frac{t}{2}\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - \frac{t}{2}$$

$$= \frac{t}{2i}\cot\frac{t}{2i} - \frac{t}{2}$$

$$= 1 - \frac{t}{2} + \frac{B_1t^2}{2!} - \frac{B_2t^4}{4!} + \dots.$$

Hence

$$\sum_{n=1}^\infty \frac{\phi_n(z)\,t^n}{n!} = \left\{zt + \frac{z^2t^2}{2!} + \frac{z^3t^3}{3!} + \dots\right\}\left\{1 - \frac{t}{2} + \frac{B_1t^2}{2!} - \frac{B_2t^4}{4!} + \dots\right\}.$$

From this, by equating coefficients of t^n, we have

$$\phi_n(z) = z^n - \frac{n}{2}z^{n-1} + \binom{n}{2}B_1z^{n-2} - \binom{n}{4}B_2z^{n-4} + \binom{n}{6}B_3z^{n-6}\dots,$$

the last term being that in z or z^2; this is the explicit expression of the nth Bernoullian polynomial.

The Bernoullian numbers and polynomials were introduced into analysis by Jacob Bernoulli in 1713.

Example. Shew that

$$\phi_n(z) = (-1)^n \phi_n(1-z).$$

64. *The Maclaurin-Bernoullian expansion.*

In Darboux's formula write $\phi(t) = \phi_n(t)$, where $\phi_n(t)$ is the nth Bernoullian polynomial.

Now from the equation

$$\phi_n(t+1) - \phi_n(t) = nt^{n-1},$$

we have by differentiating k times

$$\phi_n^{(k)}(t+1) - \phi_n^{(k)}(t) = n(n-1)\dots(n-k)\,t^{n-k-1}.$$

Putting $t=0$ in this, we have

$$\phi_n^{(k)}(1) = \phi_n^{(k)}(0).$$

But the value of $\phi_n^{(k)}(0)$ is obtained by comparing the expansion

$$\phi_n(z) = \phi_n(0) + z\phi_n'(0) + \frac{z^2}{2!}\phi_n''(0) + \dots$$

with the expansion

$$\phi_n(z) = z^n - \frac{n}{2}z^{n-1} + \binom{n}{2}B_1 z^{n-2} - \binom{n}{4}B_2 z^{n-4} + \dots.$$

Substituting the values of $\phi_n^{(k)}(1)$ and $\phi_n^{(k)}(0)$ thus obtained in Darboux's result, we find what is known as the *Maclaurin-Bernoullian formula,*

$$\begin{aligned}(z-a)f'(a) = f(z) - f(a) - \frac{z-a}{2}\{f'(z) - f'(a)\} \\ + \frac{B_1(z-a)^2}{2!}\{f''(z) - f''(a)\} + \dots \\ + \frac{(-1)^n B_{n-1}(z-a)^{2n-2}}{(2n-2)!}\{f^{(2n-2)}(z) - f^{(2n-2)}(a)\} \\ - \frac{(z-a)^{2n+1}}{2n!}\int_0^1 \phi_{2n}(t) f^{(2n+1)}\{a + (z-a)t\}\,dt.\end{aligned}$$

In certain cases the last term tends to zero as n tends to infinity, and we can thus derive an infinite series from the formula.

Example. If $f(z)$ be an odd function of z, shew that

$$\begin{aligned}zf'(z) = f(z) + \frac{B_1}{2!}(2z)^2 f''(z) + \dots + (-1)^n \frac{B_{n-1}(2z)^{2n-2}}{(2n-2)!} f^{(2n-2)}(z) \\ - \frac{2^{2n}z^{2n+1}}{2n!}\int_0^1 \phi_{2n}(t) f^{(2n+1)}(-z+2zt)\,dt,\end{aligned}$$

where $\phi_n(t)$ is the Bernoullian polynomial of order n.

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65. *Burmann's theorem.*

We shall next consider a number of theorems which have for their object *the expansion of one function in powers of another function.*

Let $\phi(z)$ be a function of z, which takes the value b when z takes the value a, so that

$$b = \phi(a).$$

Suppose that $\phi(z)$ is an analytic function of z, regular in the neighbourhood of the value $z = a$, and that $\phi'(a)$ is not zero. Then Taylor's theorem furnishes the expansion

$$\phi(z) - b = \phi'(a)(z-a) + \frac{\phi''(a)}{2!}(z-a)^2 + \dots,$$

and on reversing this series we obtain

$$z - a = \frac{1}{\phi'(a)}\{\phi(z) - b\} - \frac{1}{2}\frac{\phi''(a)}{\{\phi'(a)\}^3}\{\phi(z) - b\}^2 + \dots,$$

which expresses z as a regular function of the variable $\{\phi(z) - b\}$, for values of z in the neighbourhood of a. If then $f(z)$ be a regular function of z in the neighbourhood of a, it follows therefore that $f(z)$ is a regular function of $\{\phi(z) - b\}$ in this neighbourhood, and so an expansion of the form

$$f(z) = f(a) + a_1\{\phi(z) - b\} + \frac{a_2}{2!}\{\phi(z) - b\}^2 + \frac{a_3}{3!}\{\phi(z) - b\}^3 + \dots$$

will exist, which, as it is a power-series in $\{\phi(z) - b\}$, will be valid so long as

$$|\phi(z) - b| < r,$$

where r is some constant.

The actual expansion is given by the following theorem, which is generally known as *Burmann's theorem.*

If $\psi(z)$ be a function of z defined by the equation

$$\psi(z) = \frac{z-a}{\phi(z) - b},$$

then the function $f(z)$ can for a certain domain of values of z be expanded in the form

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{\{\phi(z) - b\}^n}{n!} \frac{d^{n-1}}{da^{n-1}}[f'(a)\{\psi(a)\}^n];$$

and the remainder after n terms in the series is

$$\frac{1}{2\pi i}\int_a^z \int_\gamma \left[\frac{\phi(z) - b}{\phi(t) - b}\right]^{n-1} \frac{f'(t)\,\phi'(z)\,dt\,dz}{\phi(t) - \phi(z)},$$

where γ is a simple contour in the t-plane, enclosing the point $t = a$.

To prove this, we have

$$f(z)-f(a)=\int_a^z f'(z)\,dz=\frac{1}{2\pi i}\int_a^z\int_\gamma \frac{f'(t)\,\phi'(z)\,dt\,dz}{\phi(t)-\phi(z)}$$

$$=\frac{1}{2\pi i}\int_a^z\int_\gamma \frac{f'(t)\,\phi'(z)\,dt\,dz}{\phi(t)-b}\left[1+\frac{\phi(z)-b}{\phi(t)-b}+\dots\right.$$

$$\left.+\frac{\{\phi(z)-b\}^{n-1}}{\{\phi(t)-b\}^{n-2}\{\phi(t)-\phi(z)\}}\right].$$

But $\displaystyle \frac{1}{2\pi i}\int_a^z\int_\gamma\left[\frac{\phi(z)-b}{\phi(t)-b}\right]^k\frac{f'(t)\,\phi'(z)\,dt\,dz}{\phi(t)-b}=\frac{\{\phi(z)-b\}^{k+1}}{2\pi i\,(k+1)}\int_\gamma\frac{f'(t)\,dt}{\{\phi(t)-b\}^{k+1}}$

$$=\frac{\{\phi(z)-b\}^{k+1}}{2\pi i\,(k+1)}\int_\gamma\frac{f'(t)\{\psi(t)\}^{k+1}\,dt}{(t-a)^{k+1}}=\frac{\{\phi(z)-b\}^{k+1}}{2\pi i\,(k+1)!}\frac{d^k}{da^k}[f'(a)\,\{\psi(a)\}^{k+1}].$$

Therefore $\displaystyle f(z)=f(a)+\sum_{k=1}^{k=n-1}\frac{\{\phi(z)-b\}^k}{2\pi i k!}\frac{d^{k-1}}{da^{k-1}}[f'(a)\,\{\psi(a)\}^k]$

$$+\frac{1}{2\pi i}\int_a^z\int_\gamma\left[\frac{\phi(z)-b}{\phi(t)-b}\right]^{n-1}\frac{f'(t)\,\phi'(z)\,dt\,dz}{\phi(t)-\phi(z)}.$$

Example 1. Prove that

$$z=a+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}C_n(z-a)^n e^{n(z^2-a^2)}}{n!},$$

where

$$C_n=(2na)^{n-1}-\frac{n(n-1)(n-2)}{1!}(2na)^{n-3}+\frac{n^2(n-1)(n-2)(n-3)(n-4)}{2!}(2na)^{n-5}\dots.$$

To obtain this expansion, write

$$f(z)=z,\quad \phi(z)-b=(z-a)\,e^{z^2-a^2},\quad \psi(z)=e^{a^2-z^2},$$

in the above expression of Burmann's theorem; we thus have

$$z=a+\sum_{n=1}^{\infty}\frac{1}{n!}(z-a)^n e^{n(z^2-a^2)}\left\{\frac{d^{n-1}}{dz^{n-1}}e^{n(a^2-z^2)}\right\}_{z=a}.$$

But

$$\left\{\frac{d^{n-1}}{dz^{n-1}}e^{n(a^2-z^2)}\right\}_{z=a}=\left\{\frac{d^{n-1}}{dt^{n-1}}e^{-n(2at+t^2)}\right\}_{t=0}\qquad\text{(putting } z=a+t\text{)}$$

$$=(n-1)!\times\text{coefficient of } t^{n-1}\text{ in the expansion of } e^{-nt(2a+t)}$$

$$=(n-1)!\times\text{coefficient of } t^{n-1}\text{ in }\sum_{r=0}^{\infty}\frac{(-1)^r n^r t^r(2a+t)^r}{r!}$$

$$=(n-1)!\times\sum_{r=0}^{n-1}\frac{(-1)^r n^r(2a)^{2r-n+1}}{(n-1-r)!\,(2r-n+1)!}.$$

The highest value of r which gives a term in the summation is $r=n-1$. Arranging therefore the summation in descending indices r, beginning with $r=n-1$, we have

$$\left\{\frac{d^{n-1}}{dz^{n-1}}e^{n(a^2-z^2)}\right\}_{z=a}=(-1)^{n-1}\left\{(2na)^{n-1}-\frac{n(n-1)(n-2)}{1!}(2na)^{n-3}+\dots\right\}$$

$$=(-1)^{n-1}C_n,$$

which gives the required result.

Example 2. Obtain the expression

$$z^2 = \sin^2 z + \frac{2}{3} \cdot \frac{1}{2} \sin^4 z + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{3} \sin^6 z + \dots.$$

Example 3. Let a line p be drawn through the origin in the z-plane, perpendicular to the line which joins the origin to any point a. If z be any point on the z-plane which is on the same side of the line p as the point a is, shew that

$$\log z = \log a + 2 \sum_{m=1}^{\infty} \frac{1}{2m+1} \left(\frac{z-a}{z+a}\right)^{2m+1}.$$

66. *Teixeira's extended form of Burmann's theorem.*

In the last paragraph we have not investigated closely the conditions of convergence of Burmann's series, for the reason that the theorem itself will next be stated in a much more general form, which bears the same relation to the theorem just given that Laurent's theorem bears to Taylor's series: viz., in the last paragraph we were concerned only with the expansion of a function in *positive* powers of another function, whereas we shall now discuss the expansion of a function in *positive and negative* powers of the second function.

The general statement of the theorem is due to Teixeira*, whose exposition we shall follow in the next two paragraphs.

Suppose (1) that $f(z)$ is a regular function of z in a ring-shaped region A, bounded by an outer curve S and an inner curve s; (2) that $\theta(z)$ is a regular function everywhere inside S, and has a single zero a within this contour; (3) that x is the affix of some point within A; (4) that for all points of the contour S we have

$$|\theta(x)| < |\theta(z)|,$$

and for all points of the contour s we have

$$|\theta(x)| > |\theta(z)|.$$

The equation

$$\theta(z) - \theta(x) = 0$$

has, in this case, a single root $z = x$ in the interior of S, as is seen from the equation

$$\frac{1}{2\pi i} \int_S \frac{\theta'(z)\, dz}{\theta(z) - \theta(x)} = \frac{1}{2i\pi} \left[\int_S \frac{\theta'(z)}{\theta(z)} dz + \theta(x) \int_S \frac{\theta'(z)}{\theta^2(z)} dz + \dots \right]$$
$$= \frac{1}{2\pi i} \int_S \frac{\theta'(z)\, dz}{\theta(z)},$$

of which the left-hand and right-hand members represent respectively the number of roots of the equation considered and that of the roots of the equation $\theta(z) = 0$ contained within S.

* *Crelle's Journal*, CXXII. (1900), pp. 97—123.

Cauchy's theorem therefore gives

$$f(x)=\frac{1}{2i\pi}\left[\int_S \frac{f(z)\theta'(z)\,dz}{\theta(z)-\theta(x)}-\int_s \frac{f(z)\theta'(z)\,dz}{\theta(z)-\theta(x)}\right].$$

The integrals in this formula can, as in Laurent's theorem, be expanded in powers of $\theta(x)$, by the formulae

$$\int_S \frac{f(z)\theta'(z)dz}{\theta(z)-\theta(x)}=\sum_{n=0}^{\infty}\theta^n(x)\int_S \frac{f(z)\theta'(z)dz}{\theta^{n+1}(z)},$$

$$\int_s \frac{f(z)\theta'(z)dz}{\theta(z)-\theta(x)}=-\sum_{n=1}^{\infty}\frac{1}{\theta^n(x)}\int_s f(z)\,\theta^{n-1}(z)\,\theta'(z)\,dz.$$

We thus have the formula

$$f(x)=\sum_{n=0}^{\infty}A_n\,\theta^n(x)+\sum_{n=1}^{\infty}\frac{B_n}{\theta^n(x)},$$

where

$$A_n=\frac{1}{2i\pi}\int_S \frac{f(z)\,\theta'(z)\,dz}{\theta^{n+1}(z)},$$

$$B_n=\frac{1}{2i\pi}\int_s f(z)\,\theta^{n-1}(z)\,\theta'(z)\,dz.$$

This gives a development of $f(x)$ according to positive and negative powers of $\theta(x)$, valid for all points x within the ring-shaped space A.

67. *Evaluation of the coefficients.*

If the function $f(z)$ has no singularities but poles in the region limited by the curve s, the integrals which occur in the preceding formula can be evaluated in the following way.

Let $b_1, b_2, \ldots b_k$ be the poles; and let $c_1, c_2, \ldots c_k$, c, be circles with centres $b_1, b_2, \ldots b_k$, a, respectively, and with very small radii.

Then

$$A_n=\frac{1}{2\pi i}\int_S \frac{f(z)\,\theta'(z)\,dz}{\theta^{n+1}(z)}=\frac{1}{2\pi i}\int_S \frac{f'(z)\,dz}{n\theta^n(z)}$$

$$=\sum_{m=1}^{k}\frac{1}{2ni\pi}\int_{c_m}\frac{f'(z)\,dz}{\theta^n(z)}+\frac{1}{2ni\pi}\int_c \frac{f'(z)\,dz}{\theta^n(z)},$$

and

$$B_n=\frac{1}{2\pi i}\int_s f(z)\,\theta^{n-1}(z)\,\theta'(z)\,dz$$

$$=-\frac{1}{2ni\pi}\int_s f'(z)\,\theta^n(z)\,dz$$

$$=-\frac{1}{2ni\pi}\sum_{m=1}^{k}\int_{c_m}f'(z)\,\theta^n(z)\,dz.$$

Thus if α_m be the degree of multiplicity of the pole b_m, and if $\dfrac{\theta(x)}{x-a}$ be denoted by $\theta_1(x)$, we have

$$A_n = \frac{1}{n!}\left[\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{f'(x)}{\theta_1{}^n(x)}\right\}\right]_{x=a} + \sum_{m=1}^{k} \frac{1}{\alpha_m!\,n}\left[\frac{d^{\alpha_m}}{dx^{\alpha_m}}\left\{\frac{f'(x)(x-b_m)^{\alpha_m+1}}{\theta^n(x)}\right\}\right]_{x=b_m},$$

and
$$B_n = -\sum_{m=1}^{k} \frac{1}{\alpha_m!\,n}\left[\frac{d^{\alpha_m}}{dx^{\alpha_m}}\{f'(x)\,\theta^n(x)\,(x-b_m)^{\alpha_m+1}\}\right]_{x=b_m}.$$

It may happen that a is also a pole of $f(x)$. It is easily seen that in this case A_n is given by the formula

$$A_n = \sum_{m=1}^{k} \frac{1}{\alpha_m!\,n}\left[\frac{d^{\alpha_m}}{dx^{\alpha_m}}\left\{\frac{f(x)(x-b_m)^{\alpha_m+1}}{\theta^n(x)}\right\}\right]_{x=b_m} + \frac{1}{(n+\beta)!\,n}\left[\frac{d^{\beta+n}}{dx^{\beta+n}}\left\{\frac{f'(x)(x-a)^{\beta+1}}{\theta_1{}^n(x)}\right\}\right]_{x=a},$$

where β is the degree of multiplicity of the pole a; the formula for B_n must likewise be replaced by

$$B_n = -\sum_{m=1}^{k} \frac{1}{\alpha_m!\,n}\left[\frac{d^{\alpha_m}}{dx^{\alpha_m}}\{f'(x)\theta^n(x)(x-b_m)^{\alpha_m+1}\}\right]_{x=b_m} - \frac{1}{(\beta-n)!\,n}\left[\frac{d^{\beta-n}}{dx^{\beta-n}}\{f'(x)\,\theta_1{}^n(x)\,(x-a)^{\beta+1-n}\}\right]_{x=a},$$

when
$$n \leqslant \beta.$$

The preceding formulae do not give the value of A_0; this can be found from the formula

$$A_0 = \sum_{m=1}^{k} \frac{1}{2i\pi}\int_{c_m} \frac{f(z)\,\theta'(z)\,dz}{\theta(z)} + \frac{1}{2i\pi}\int_{c} \frac{f(z)\,\theta'(z)\,dz}{\theta(z)},$$

which gives

$$A_0 = \sum_{m=1}^{k} \frac{1}{(\alpha_m-1)!}\left[\frac{d^{\alpha_m-1}}{dx^{\alpha_m-1}}\left\{\frac{f(x)\,\theta'(x)\,(x-b_m)^{\alpha_m}}{\theta(x)}\right\}\right]_{x=b_m} + f(a),$$

when a is a regular point for $f(x)$; and

$$A_0 = \sum_{m=1}^{k} \frac{1}{(\alpha_m-1)!}\left[\frac{d^{\alpha_m-1}}{dx^{\alpha_m-1}}\left\{\frac{f(x)\,\theta'(x)\,(x-b_m)^{\beta}}{\theta(x)}\right\}\right]_{x=b_m} + \frac{1}{\beta!}\left[\frac{d^{\beta}}{dx^{\beta}}\left\{\frac{f(x)\,\theta'(x)\,(x-a)^{\beta}}{\theta_1(x)}\right\}\right]_{x=a},$$

when a is a pole of $f(x)$.

Example 1. Shew that

$$x=\frac{1}{2}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2.4}\left(\frac{2x}{1+x^2}\right)^3+\frac{1.3}{2.4.6}\left(\frac{2x}{1+x^2}\right)^5+\dots,$$

when $-1 < x < 1$.

Shew that the second member represents $\frac{1}{x}$, when $|x| > 1$.

Example 2. If $S_{2n}^{(m)}$ denote the sum of all combinations of the numbers

$$2^2,\ 4^2,\ 6^2, \dots (2n-2)^2,$$

taken m together, shew that

$$\frac{1}{z}=\frac{1}{\sin z}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}\left\{\frac{1}{2n+3}-\frac{S_{2(n+1)}^{(1)}}{2n+1}+\dots+\frac{S_{2(n+1)}^{(n)}}{3}\right\}(\sin z)^{2n+1},$$

the expansion being valid for all values of z represented by points within the oval whose equation is $|\sin z|=1$ and which contains the point $z=0$. (Teixeira.)

68. *Expansion of a function of a root of an equation, in terms of a parameter occurring in the equation.*

Now consider the equation

$$\theta(x)=(x-a)\,\theta_1(x)=t,$$

where t is a number such that along the contour S we have $|\theta(z)|>|t|$, and along the contour s we have $|\theta(z)|<|t|$.

The equation $\theta(x)=t$, regarded as an equation in x, will then have a single root in the ring-shaped region bounded by the curves S and s; we see, in fact, from the equations

$$\frac{1}{2\pi i}\int_S\frac{\theta'(z)\,dz}{\theta(z)-t}=\frac{1}{2\pi i}\left[\int_S\frac{\theta'(z)}{\theta(z)}\,dz+t\int_S\frac{\theta'(z)}{\theta^2(z)}\,dz+\dots\right]$$
$$=1,$$

and
$$\frac{1}{2\pi i}\int_s\frac{\theta'(z)\,dz}{\theta(z)-t}=-\frac{1}{2\pi i}\left[\frac{1}{t}\int_s\theta'(z)\,dz+\frac{1}{t^2}\int_s\theta'(z)\,\theta(z)\,dz+\dots\right]$$
$$=0,$$

that the equation in question has one root in the interior of S and none in the interior of s.

Then if the function $f(x)$ is regular in the region limited by S and s, we see from the preceding articles that the formula

$$f(x)=\sum_{n=0}^{\infty}A_nt^n+\sum_{n=1}^{\infty}\frac{B_n}{t^n},$$

where A_n and B_n have the values already found, gives the expansion in powers of t of the function $f(x)$ of the root considered.

As an example of this formula consider the equation $(x-a)\operatorname{cosec} x=t$, and let

$$f(x)=\frac{1}{x-a}.$$

Then we find

$$A_0 = -\frac{\cos a}{\sin a},$$

$$A_n = -\frac{1}{(n+1)!\,n}\frac{d^{n+1}(\sin^n a)}{da^{n+1}},$$

$$B_1 = \frac{1}{\sin a}, \quad B_2 = B_3 = \dots = 0.$$

Hence

$$\frac{1}{x-a} = -\frac{\cos a}{\sin a} - \sum_{n=1}^{\infty}\frac{t^n}{(n+1)!\,n}\frac{d^{n+1}(\sin^n a)}{da^{n+1}} + \frac{1}{t\sin a},$$

and thus gives the expansion, in ascending powers of t, of $\frac{1}{x-a}$, where x is given in terms of t by the equation

$$x = a + t\sin x. \qquad \text{(Teixeira.)}$$

69. *Lagrange's theorem.*

Suppose now that the function $f(z)$ is regular at all points in the interior of S, so that the poles $b_1, b_2, \dots b_k$ do not exist. Then the formulae which give the quantities A_n and B_n now become

$$A_n = \frac{1}{n!}\frac{d^{n-1}}{da^{n-1}}\left\{\frac{f'(a)}{\theta_1^n(a)}\right\} \qquad (n \geqslant 1),$$

$$A_0 = f(a),$$

$$B_n = 0.$$

Moreover the contour s can now be dispensed with, and the theorem of the last article takes the following form:

Let $f(z)$ be a regular function of z at all points in the interior of a contour S, and let $\theta(z)$ be a regular function with no zero in the interior of S. Let a be a point inside S, and t a number such that for all points z on S we have

$$|(z-a)\,\theta(z)| > |t|.$$

Then the equation $(z-a)\,\theta(z) = t$ will have one root x in the interior of S, and $f(x)$ will be given as a power-series in t by the expansion

$$f(x) = f(a) + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^{n-1}}{da^{n-1}}\left\{\frac{f'(a)}{\theta^n(a)}\right\}t^n.$$

This result was published by Lagrange in 1768; it is usually stated in a slightly different form, to obtain which we shall write

$$\phi(z) = \frac{1}{\theta(z)};$$

the result may now be enunciated as follows:

If $f(z)$ and $\phi(z)$ be regular functions of z within a contour S surrounding a point a, and if t be a quantity such that the inequality

$$|t\,\phi(z)| < |z-a|$$

is satisfied at all points z on the perimeter of S, then the equation

$$z = a + t\phi(z),$$

regarded as an equation in z, has one root in the interior of S: and if this root be denoted by x, then any regular function of x can be expanded as a power-series in t by the formula

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} [f'(a) \{\phi(a)\}^n].$$

This result is of course a particular case of the more general theorem given in § 68.

Example 1. Within the contour surrounding $z=a$ and defined by the inequality

$$|z(z-a)| > |a|,$$

the equation

$$z - a - \frac{a}{z} = 0$$

has one root z, the expansion of which is given by Lagrange's theorem in the form

$$z = a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!}{n!\,(n-1)!\,a^{2n-1}} a^n.$$

Now from the ordinary theory of quadratic equations, we know that the equation

$$z - a - \frac{a}{z} = 0$$

has two roots, namely

$$\frac{a}{2}\left\{1 + \sqrt{1 + \frac{4a}{a^2}}\right\} \text{ and } \frac{a}{2}\left\{1 - \sqrt{1 + \frac{4a}{a^2}}\right\};$$

and our expansion *represents the former of these only*—an example of the need for care in the discussion of these series. If however we regard the expansion as a power-series in a, and derive other power-series from it by continuation in the a-plane, we shall ultimately arrive at the series

$$z = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{n!\,(n-1)!} \frac{a^n}{a^{2n-1}},$$

which represents the other branch of the function z.

Example 2. If y be that one of the roots of the equation

$$y = 1 + zy^2$$

which reduces to unity when z is zero, shew that

$$y^n = 1 + nz + \frac{n(n+3)}{2!} z^2 + \frac{n(n+4)(n+5)}{3!} z^3$$

$$+ \frac{n(n+5)(n+6)(n+7)}{4!} z^4 + \frac{n(n+6)(n+7)(n+8)(n+9)}{5!} z^5 + \ldots$$

so long as $|z| < \frac{1}{4}$.

Example 3. If x be that one of the roots of the equation

$$x = 1 + yx^a$$

which reduces to unity when y is zero, shew that

$$\log x = y + \frac{2a-1}{2} y^2 + \frac{(3a-1)(3a-2)}{2\,.\,3} y^3 + \ldots,$$

the expansion being valid so long as

$$|y| < |(a-1)^{a-1} a^{-a}|. \qquad \text{(McClintock.)}$$

70. *Rouché's extension of Lagrange's theorem.*

Consider now two functions $f(z)$ and $\phi(z)$, which are regular at all points within a contour C, on the perimeter of which the inequality $\left|\frac{\alpha\phi(z)}{f(z)}\right| < 1$ is satisfied.

Then we shall shew that *if the equation $f(z)=0$ have p roots $a_1, a_2, \ldots a_p$ in the region contained by C, the equation $f(z) - \alpha\phi(z) = 0$ will have p roots $a_1', a_2', \ldots a_p'$, in that region; and for every function $F(z)$ regular in the region we shall have*

$$\sum_{r=1}^{p} F(a_r') = \sum_{r=1}^{p} F(a_r) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \sum_{r=1}^{p} \frac{d^{n-1}}{da_r^{\,n-1}} \left\{ \frac{F'(a_r)\,\overline{\phi(a_r)|}^{\,n}}{\overline{\psi(a_r)|}^{\,n}} \right\},$$

where
$$\psi(z) = \frac{f(z)}{z-a_r}.$$

We may note that this theorem reduces to that of Lagrange when

$$f(z) = z - a \text{ and } p = 1.$$

The result stated may be obtained in the following way:

We have
$$\Sigma F(a_r') = \frac{1}{2\pi i} \int_C F(z) \frac{f'(z) - \alpha\phi'(z)}{f(z) - \alpha\phi(z)} dz$$

$$= \frac{1}{2\pi i} \int_C F(z)\, dz \,\{f'(z) - \alpha\phi'(z)\} \left[\frac{1}{f(z)} + \frac{\alpha\phi(z)}{\{f(z)\}^2} + \ldots + \frac{\alpha^{n-1}\{\phi(z)\}^{n-1}}{\{f(z)\}^n} \right.$$
$$\left. + \frac{\alpha^n \{\phi(z)\}^n}{\{f(z)\}^n \{f(z) - \alpha\phi(z)\}} \right]$$

$$= \frac{1}{2\pi i} \int_C F(z)\, dz \left[\frac{f'(z)}{f(z)} - \alpha \frac{d}{dz}\left\{\frac{\phi(z)}{f(z)}\right\} - \ldots - \frac{\alpha^{n-1}}{n-1} \frac{d}{dz} \left\{\frac{\phi(z)}{f(z)}\right\}^{n-1} \right.$$
$$\left. - \alpha^n \left\{\frac{\phi(z)}{f(z)}\right\}^{n-1} \frac{\phi'(z)}{f(z)} + \alpha^n \left\{\frac{\phi(z)}{f(z)}\right\}^n \frac{f'(z) - \alpha\phi'(z)}{f(z) - \alpha\phi(z)} \right].$$

When n is large, the last integral tends to zero: we thus have on the right-hand side a power-series in α, in which the coefficient of α^n is

$$-\frac{1}{2\pi i n} \int_C F(z)\, dz \,.\, \frac{d}{dz} \left\{\frac{\phi(z)}{f(z)}\right\}^n \text{ or } \frac{1}{2\pi i n} \int_C F'(z) \left\{\frac{\phi(z)}{f(z)}\right\}^n dz,$$

or
$$\sum_{r=1}^{p} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left\{ \frac{F'(z)\,\{\phi(z)\}^n (z-a_r)^m}{\{f(z)\}^n} \right\} \right]_{z=a_r},$$

or
$$\frac{1}{n!}\sum_{r=1}^{p}\frac{d^{n-1}}{da_r^{\,n-1}}\left[\frac{F'(a_r)\,\overline{\phi(a_r)|}^{\,n}}{\psi(a_r)|^{\,n}}\right],$$

which establishes the theorem. Putting $F(z)=1$, it is seen that the number of roots a' is p.

71. Teixeira has published the following generalisation of Lagrange's theorem, the proof of which may be left to the student. Let

$$z=t+x\phi_1(z)+x^2\phi_2(z)+\ldots+x^k\phi_k(z),$$

where $\phi_1(z), \ldots \phi_k(z)$ are regular functions of z in the interior of a contour K, and t is a point inside K. Let a be a positive quantity, so small that the condition

$$\left|\frac{a\phi_1(z)}{z-t}\right|+\left|\frac{a^2\phi_2(z)}{z-t}\right|+\ldots+\left|\frac{a^k\phi_k(z)}{z-t}\right|<1$$

is satisfied along the contour K. Then to every value of x which satisfies the condition $|x|<a$ there corresponds a unique value of z in the interior of K; and $f(z)$, where f is a regular function at all points in the interior of K, can be expanded in ascending powers of x by the formula

$$f(z)=f(t)+xf'(t)\,\phi(t)+\ldots+x^n\Sigma\,\frac{d^{b-1}[f'(t)\,\{\phi_1(t)\}^{a}\,\{\phi_2(t)\}^{\beta}\ldots\{\phi_k(t)\}^{\lambda}]}{a!\,\beta!\,\gamma!\ldots\lambda!\,dt^{b-1}},$$

where the summation is extended over all positive integral solutions of the equation

$$a+2\beta+3\gamma+\ldots+k\lambda=n,$$

and where

$$b=a+\beta+\gamma+\ldots+\lambda.$$

Another form of this result is

$$f(z)=f(t)+\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{x^{\mu}}{(\nu+1)!}\frac{d^{\nu}}{dt^{\nu}}\{f'(t)\,\phi_{\nu-1,\mu}(t)\},$$

where the quantities $\phi_{\nu,\mu}$ are obtained from the equations

$$\phi_{1,\mu}=\phi_{\mu},$$
$$\phi_{\nu+1,\mu}=\phi_{1,0}\phi_{\nu,\mu}+\phi_{1,1}\phi_{\nu,\mu+1}+\ldots+\phi_{1,\mu}\phi_{\nu,\mu}.$$

72. *Laplace's extension of Lagrange's theorem.*

Lagrange's result can easily be extended to a case in which the given equation is of a somewhat more general type.

Suppose that the equation

$$z=\psi\{a+t\,\phi(z)\}$$

is given, and that it is desired to expand some function $f(z)$ of a root of this equation in ascending powers of t.

If we write $\qquad a+t\phi(z)=u,$

the equation reduces to $\qquad u=a+t\phi\{\psi(u)\}.$

The problem of expanding $f(z)$ is therefore equivalent to that of expanding $f\{\psi(u)\}$, where u is given by the last equation; and this can be done by Lagrange's theorem.

73. *A further generalisation of Taylor's theorem.*

The series of Laurent, Darboux, Burmann, etc. may be regarded as extensions in different directions of the fundamental series of Taylor. A generalisation of Taylor's theorem of a somewhat different character to these, is furnished by the following result, the proof of which may be left to the student.

If $f(z)$ and $\theta(z)$ are regular functions of z in the neighbourhood of the point $z=x$, and if

$$\theta_1(z)=\int_x^z \theta(t)\,dt, \qquad \theta_2(z)=\int_x^z \theta_1(t)\,dt,$$

and generally

$$\theta_n(z)=\int_x^z \theta_{n-1}(t)\,dt,$$

then, for values of z in the neighbourhood of the point x, $f(z)$ can be expanded in a series of the form

$$f(z)=a_0\theta(z)+a_1\theta_1(z)+a_2\theta_2(z)+\ldots+a_n\theta_n(z)+\ldots,$$

where

$$a_0=\frac{f(x)}{\theta(x)},$$

$$a_1=\frac{d}{dx}\left\{\frac{f(x)}{\theta(x)}\right\},$$

and generally

$$a_n=\frac{1}{\{\theta(x)\}^{n-1}}\frac{d}{dx}\left[\theta(x)\frac{d}{dx}\left\{\theta(x)\frac{d}{dx}\left(\ldots\left[\theta(x)\frac{d}{dx}\left\{\frac{f(x)}{\theta(x)}\right\}\right]\right)\right\}\right],$$

the number of differentiations in the last expression being n.

It is clear that Taylor's series is obtained from this expansion by putting $\theta(z)=1$.

Example 1. Shew that

$$\theta_n(z)=\int_x^z \frac{(z-t)^{n-1}}{(n-1)!}\theta(t)\,dt.$$

Example 2. Shew that

$$a_n=\frac{da_{n-1}}{dx}+(n-1)a_{n-1}\frac{\theta'(x)}{\theta(x)}.$$

(Laurent, *Journ. Math. Spéc.*, 1897.)

Example 3. By writing $\theta(z)=e^z$, obtain the expansion of an arbitrary function of z in a series of the form

$$a_0e^{z-x}+a_1(e^{z-x}-1)+\ldots+a_n\left\{e^{z-x}-1-(z-x)\ldots-\frac{(z-x)^{n-1}}{(n-1)!}\right\}+\ldots,$$

where a_0, a_1 are independent of z.

Example 4. In the general result, shew that when $x=0$ we have

$$\Sigma a_nt^n\times\Sigma B_nt^n=\Sigma A_nt^n,$$

where

$$f(z)=\Sigma\frac{A_n}{n!}z^n \text{ and } \theta(z)=\Sigma\frac{B_n}{n!}z^n.$$

(Guichard, *Annales de l'Éc. Norm.*, 1887.)

74. *The expansion of a function in rational functions.*

Consider now a function $f(z)$, whose only singularities in the finite part of the plane are simple poles $a_1, a_2, a_3, \ldots$: let $c_1, c_2, \ldots$ be the residues at these poles, and let C be a circle of very large radius R not passing through any poles, so that $f(z)$ is finite at all points in the circumference of C. (The function $\operatorname{cosec} z$ may be cited as an example of the class of functions considered.) Suppose further that at all points on the circumference of C, the modulus of $f(z)$ is less than M, where M is a quantity which remains finite when large values of R are taken.

Then $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-x}\,dz =$ sum of residues of $\frac{f(z)}{z-x}$ at points in the interior of C

$$= f(x) + \sum_r \frac{c_r}{a_r - x},$$

where the summation extends over all poles in the interior of C.

But
$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-x}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z} + \frac{x}{2\pi i}\int_C \frac{f(z)}{z(z-x)}\,dz$$
$$= f(0) + \sum_n \frac{c_n}{a_n} + \frac{x}{2\pi i}\int_C \frac{f(z)\,dz}{z(z-x)},$$

if we suppose the function $f(z)$ to be regular at the origin.

Now R being supposed large, $\int_C \frac{f(z)\,dz}{z(z-x)}$ is of the order $\frac{1}{R}$ of small quantities, and so tends to zero as R tends to infinity.

Therefore on making R infinitely great, we have

$$0 = f(x) - f(0) + \sum_n c_n \left(\frac{1}{a_n - x} - \frac{1}{a_n}\right),$$

or

$$f(x) = f(0) + \sum_n c_n \left\{\frac{1}{x - a_n} + \frac{1}{a_n}\right\},$$

which is an expansion of $f(x)$ in rational functions of x.

If instead of the condition $|f(z)| < M$ we have the condition $|f(z)| < MR^n$, where M is finite for all values of R and n is a positive integer, then we should have to expand $\int_C \frac{F(z)\,dz}{z-x}$ by writing

$$\frac{1}{z-x} = \frac{1}{z} + \frac{x}{z^2} + \ldots + \frac{x^{n+1}}{z^{n+1}(z-x)},$$

and should obtain a similar but somewhat more complicated expansion.

Example 1. Prove that

$$\operatorname{cosec} z = \frac{1}{z} + \Sigma(-1)^n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right),$$

the summation extending to all positive and negative values of n.

To obtain this result, let $\operatorname{cosec} z - \frac{1}{z} = f(z)$. The singularities of this function are at the points $z = n\pi$, where n is any positive or negative integer.

For points near one of these singularities, put $z=n\pi+\zeta$. Then

$$f(z)=\operatorname{cosec}(n\pi+\zeta)-\frac{1}{n\pi+\zeta}=\frac{(-1)^n}{\sin\zeta}-\frac{1}{n\pi}\left(1+\frac{\zeta}{n\pi}\right)^{-1}$$

$$=\frac{(-1)^n}{\zeta}\left(1-\frac{\zeta^2}{3\,!}+\frac{\zeta^4}{5\,!}\ldots\right)^{-1}-\frac{1}{n\pi}\left(1+\frac{\zeta}{n\pi}\right)^{-1}$$

$$=\frac{(-1)^n}{z-n\pi}-\frac{1}{n\pi}+\text{positive powers of }\zeta.$$

The residue of $f(z)$ at the singularity $n\pi$ is therefore $(-1)^n$. Applying now the general theorem

$$f(z)=f(0)+\Sigma c_n\left[\frac{1}{z-a_n}+\frac{1}{a_n}\right],$$

where c_n is the residue at the singularity a_n, we have

$$f(z)=f(0)+\Sigma(-1)^n\left\{\frac{1}{z-n\pi}+\frac{1}{n\pi}\right\}.$$

But

$$f(0)=\operatorname{Lt}_{z=0}\left[\frac{1}{z}+(\text{positive powers of }z)-\frac{1}{z}\right]=0.$$

Therefore

$$\operatorname{cosec} z=\frac{1}{z}+\Sigma(-1)^n\left[\frac{1}{z-n\pi}+\frac{1}{n\pi}\right],$$

which is the required result.

Example 2. If a is real and positive and less than unity, shew that

$$\frac{e^{az}}{e^z-1}=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z\cos 2na\pi-4n\pi\sin 2na\pi}{z^2+4n^2\pi^2}.$$

For if $f(z)=\dfrac{e^{az}}{e^z-1}-\dfrac{1}{z}$, the singularities of $f(z)$ are at the points $z=2n\pi i$, where

$$n=\pm1,\ \pm2,\ \pm3,\ \ldots\ \pm\infty.$$

For points z near $z=2n\pi i$, put $z=2n\pi i+\zeta$. Then

$$f(z)=\frac{e^{2na\pi i}e^{a\zeta}}{e^{\zeta}-1}-\frac{1}{2n\pi i+\zeta}=\frac{e^{2na\pi i}}{\zeta}(1+a\zeta+\ldots)\left(1+\frac{\zeta}{2}+\ldots\right)^{-1}-\frac{1}{2n\pi i}\left(1+\frac{\zeta}{2n\pi i}\right)^{-1}$$

$$=\frac{e^{2na\pi i}}{\zeta}+\text{a series of positive powers of }\zeta.$$

The residue at $z=2n\pi i$ is therefore $e^{2na\pi i}$.

Also

$$f(0)=\left[\frac{1+az+\ldots}{z+\frac{z^2}{2}+\ldots}-\frac{1}{z}\right]_{z=0}$$

$$=\left[\frac{1}{z}(1+az+\ldots)\left(1+\frac{z}{2}+\ldots\right)^{-1}-\frac{1}{z}\right]_{z=0}$$

$$=a-\tfrac{1}{2}.$$

Applying the general theorem

$$f(z)=f(0)+\Sigma c_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right),$$

we have therefore

$$\frac{e^{az}}{e^z-1}-\frac{1}{z}=a-\tfrac{1}{2}+\sum_{n=\pm1}^{\pm\infty}e^{2ni\pi a}\left(\frac{1}{z-2ni\pi}+\frac{1}{2n\pi i}\right)$$

$$=a-\tfrac{1}{2}+\sum_{n=\pm1}^{\pm\infty}\frac{e^{2ni\pi a}}{z-2ni\pi}+\sum_{n=1}^{\infty}\frac{\sin 2na\pi}{n\pi}.$$

But

$$\sum_{n=1}^{\infty} \frac{\sin 2na\pi}{n\pi} = -\frac{1}{2\pi i}\log(1-e^{2a\pi i}) + \frac{1}{2\pi i}\log(1-e^{-2a\pi i})$$

$$= \frac{1}{2\pi i}\log(-e^{-2a\pi i}) = \frac{1}{2\pi i}(\pi i - 2a\pi i)$$

$$= \frac{1}{2} - a.$$

Thus

$$\frac{e^{az}}{e^z-1} - \frac{1}{z} = \sum_{n=\pm 1}^{\pm\infty} \frac{e^{2ni\pi a}}{z-2ni\pi} = \sum_{n=1}^{\infty}\left(\frac{e^{2ni\pi a}}{z-2ni\pi} + \frac{e^{-2ni\pi a}}{z+2ni\pi}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2z\cos 2na\pi - 4n\pi \sin 2na\pi}{z^2+4n^2\pi^2}.$$

Example 3. Prove that

$$\frac{1}{\pi x^2(e^x - 2\cos x + e^{-x})} = \frac{1}{2\pi x^4} - \frac{1}{e^{\pi}-e^{-\pi}}\frac{1}{\pi^4+\frac{1}{4}x^4} + \frac{2}{e^{2\pi}-e^{-2\pi}}\frac{1}{(2\pi)^4+\frac{1}{4}x^4}$$

$$- \frac{3}{e^{3\pi}-e^{-3\pi}}\frac{1}{(3\pi)^4+\frac{1}{4}x^4} + \dots.$$

For the general term of the series on the right is

$$\frac{(-1)^r r}{e^{r\pi}-e^{-r\pi}}\frac{1}{(r\pi)^4+\frac{1}{4}x^4},$$

which is the residue at either of the four singularities r, $-r$, ri, $-ri$, of the function

$$\frac{\pi z}{(\pi^4 z^4 - \frac{1}{4}x^4)(e^{\pi z}-e^{-\pi z})\sin \pi z}.$$

The singularities of this latter function which are not of the type r, $-r$, ri, $-ri$, are at the points

$$z=0, \quad z = \frac{\pm\sqrt{i}}{\sqrt{2}}\frac{x}{\pi}, \quad z = \frac{\pm\sqrt{-i}}{\sqrt{2}}\frac{x}{\pi}.$$

At $z=0$ the residue is $\dfrac{2}{\pi x^4}$;

at either of the four points $z = \dfrac{\pm\sqrt{\pm i}}{\sqrt{2}}\dfrac{x}{\pi}$, the residue is

$$\frac{\pi^2-1}{\pi \,.\, 2ix^2\left(e^{\frac{\sqrt{i}x}{\sqrt{2}}} - e^{-\frac{\sqrt{i}x}{\sqrt{2}}}\right)\sin\frac{\sqrt{i}x}{\sqrt{2}}}.$$

Therefore

$$4\sum_{r=1}^{\infty}\frac{(-1)^r r}{e^{r\pi}-e^{-r\pi}}\frac{1}{(r\pi)^4+\frac{1}{4}x^4} + \frac{2}{\pi x^4} + \frac{4i}{\pi i x^2\left(e^{\frac{\sqrt{i}x}{\sqrt{2}}} - e^{-\frac{\sqrt{i}x}{\sqrt{2}}}\right)\left(e^{i\frac{\sqrt{i}x}{\sqrt{2}}} - e^{-i\frac{\sqrt{i}x}{\sqrt{2}}}\right)}$$

$$= \frac{1}{2\pi i}\int_C \frac{\pi z\, dz}{(\pi^4 z^2 - \frac{1}{4}x^4)(e^{\pi z}-e^{-\pi z})\sin \pi z},$$

where C is an infinite contour. But at points on C, this integrand is infinitely small compared with $\dfrac{1}{z}$; the integral round C is therefore zero.

Thus
$$\frac{1}{2\pi x^4}+\sum_{r=1}^{\infty}\frac{(-1)^r r}{e^{\pi r}-e^{-\pi r}}\frac{1}{(r\pi)^4+\frac{1}{4}x^4}=\frac{-1}{\pi x^2\left(e^{\frac{\sqrt{i}x}{\sqrt{2}}}-e^{-\frac{\sqrt{i}x}{\sqrt{2}}}\right)\left(e^{i\frac{\sqrt{i}x}{\sqrt{2}}}-e^{-i\frac{\sqrt{i}x}{\sqrt{2}}}\right)}$$
$$=\frac{-1}{\pi x^2\left\{e^{(1+i)\frac{x}{2}}-e^{(-1-i)\frac{x}{2}}\right\}\left\{e^{(-1+i)\frac{x}{2}}-e^{(1-i)\frac{x}{2}}\right\}}$$
$$=\frac{-1}{\pi x^2\{e^{ix}+e^{-ix}-e^{x}-e^{-x}\}}$$
$$=\frac{1}{\pi x^2(e^{x}-2\cos x+e^{-x})},$$
which is the required result.

Example 4. Prove that
$$\sec x=4\pi\left(\frac{1}{\pi^2-4x^2}-\frac{3}{9\pi^2-4x^2}+\frac{5}{25\pi^2-4x^2}\cdots\right).$$

Example 5. Prove that
$$\operatorname{cosech} x=\frac{1}{x}-2x\left(\frac{1}{\pi^2+x^2}-\frac{1}{4\pi^2+x^2}+\frac{1}{9\pi^2+x^2}\cdots\right).$$

Example 6. Prove that
$$\operatorname{sech} x=4\pi\left(\frac{1}{\pi^2+4x^2}-\frac{3}{9\pi^2+4x^2}+\frac{5}{25\pi^2+4x^2}\cdots\right).$$

Example 7. Prove that
$$\coth x=\frac{1}{x}+2x\left(\frac{1}{\pi^2+x^2}+\frac{1}{4\pi^2+x^2}+\frac{1}{9\pi^2+x^2}+\cdots\right).$$

Example 8. Prove that
$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m^2+a^2)(n^2+b^2)}=\frac{\pi^2}{ab}\coth\pi a\coth\pi b.$$

(Cambridge Mathematical Tripos, Part I, 1899.)

75. *Expansion of a function in an infinite product.*

The theorem of the last article can be applied to the expansion of functions as infinite products.

For let $f(z)$ be a function, which has simple zeros at the points $a_1, a_2, a_3, \ldots$ where $\lim_{n=\infty}|a_n|$ is infinite; and suppose that $f(z)$ has no singularities in the finite part of the plane.

Then clearly $f'(z)$ can have no singularities in the finite part of the plane, and so $\dfrac{f'(z)}{f(z)}$ can have singularities only at the places $a_1, a_2, a_3, \ldots$.

Now for values of z near a_r, we have by Taylor's theorem

$$f(z) = (z - a_r) f'(a_r) + \frac{(z-a_r)^2}{2} f''(a_r) + \dots$$

and

$$f'(z) = f'(a_r) + (z - a_r) f''(a_r) + \dots .$$

Thus we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_r} + \text{a constant} + \text{positive powers of } (z - a_r).$$

At each of the points a_r, the function $\dfrac{f'(z)}{f(z)}$ has therefore a simple pole, with the residue $+1$.

If then $\dfrac{f'(z)}{f(z)}$ has at infinity the character of the functions considered in the last theorem, it can be expanded in the form

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{n=\infty} \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}.$$

Integrating this expression, and raising it to the exponential, we have

$$f(z) = c e^{\frac{f'(0)}{f(0)} z} \prod_{n=1}^{n=\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right\},$$

where c is a constant independent of z.

Putting $z = 0$, we see that $f(0) = c$, and thus the general result becomes

$$f(z) = f(0) e^{\frac{f'(0)}{f(0)} z} \prod_{n=1}^{n=\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right\}.$$

This furnishes the expansion, in the form of an infinite product, of any function $f(z)$ which fulfils the conditions stated.

This theorem is a case of a general theorem on the factorisation of functions, which is due to Weierstrass, and which will be found in Forsyth's *Theory of Functions*, Chapter v.

Example 1. Consider the function $f(z) = \dfrac{\sin z}{z}$, which has simple zeros at the points $r\pi$, where r is any positive or negative integer.

In this case we have $f(0) = 1, \quad f'(0) = 0,$

and so the theorem gives immediately

$$\frac{\sin z}{z} = \prod_{n=\pm 1}^{\pm\infty} \left\{ \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \right\},$$

since the condition relative to the behaviour of $\dfrac{f'(z)}{f(z)}$ at infinity is easily seen to be fulfilled.

Example 2. Prove that

$$\left\{1+\left(\frac{k}{x}\right)^2\right\}\left\{1+\left(\frac{k}{2\pi-x}\right)^2\right\}\left\{1+\left(\frac{k}{2\pi+x}\right)^2\right\}\left\{1+\left(\frac{k}{4\pi-x}\right)^2\right\}\left\{1+\left(\frac{k}{4\pi+x}\right)^2\right\}\cdots\cdots$$

$$=\frac{\cosh k-\cos x}{1-\cos x}.$$

(Trinity College Examination, 1899.)

76. *Expansion of a periodic function in a series of cotangents.*

Another mode of expansion, which may be applied to periodic functions whose poles are all simple, is that indicated in the following example.

Consider the function

$$\cot(x-a_1)\cot(x-a_2)\ldots\cot(x-a_n).$$

This is a trigonometric function of x, having poles at the points $a_1, a_2, \ldots a_n$, and also at all other points whose affixes differ from one of these quantities by a multiple of π. There is clearly no loss of generality in supposing that the real part of each of the quantities $a_1, a_2, \ldots a_n$, lies between 0 and π.

Now let $ABCD$ be a rectangle in the z-plane whose corners are the points $A\,(z=-i\infty)$, $B\,(z=\pi-i\infty)$, $C\,(z=\pi+i\infty)$, and $D\,(z=i\infty)$; and consider the integral

$$\frac{1}{2\pi i}\int\cot(z-a_1)\cot(z-a_2)\ldots\cot(z-a_n)\cot(z-x)\,dz$$

taken round the perimeter of the rectangle.

The integrals along DA and CB are equal but of opposite sign and cancel each other. Along CD, each of the cotangents has the value $-i$, so the integral along CD is $\frac{(-i)^n}{2}$. Similarly the integral along AB has the value $\frac{i^n}{2}$. The whole integral has therefore the value

$$\frac{1+(-1)^n}{2}\,i^n.$$

The singularities of the integrand in the interior of the contour are at the points $z=a_1, a_2, \ldots a_n, x$; and clearly the residue at a_r is

$$\cot(a_r-a_1)\cot(a_r-a_2)\ldots\cot(a_r-a_{r-1})\cot(a_r-a_{r+1})\ldots$$
$$\cot(a_r-a_n)\cot(a_r-x),$$

while the residue at x is

$$\cot(x-a_1)\ldots\cot(x-a_n).$$

Since the value of the integral is equal to the sum of all these residues, we thus have

$$\frac{1+(-1)^n}{2}\,i^n=\cot(x-a_1)\ldots\cot(x-a_n)+\sum_{r=1}^{r=n}\cot(a_r-a_1)\ldots$$
$$\cot(a_r-a_n)\cot(a_r-x).$$

Thus if n be even, we have

$$\cot(x-a_1)\ldots\cot(x-a_n)=\sum_{r=1}^{r=n}\cot(a_r-a_1)\ldots\cot(a_r-a_n)\cot(x-a_r)+(-1)^{\frac{n}{2}},$$

and if n be odd we have

$$\cot(x-a_1)\ldots\cot(x-a_n)=\sum_{r=1}^{r=n}\cot(a_r-a_1)\ldots\cot(a_r-a_n)\cot(x-a_r).$$

This method of decomposition into a series of cotangents is of very general application to periodic functions; it may be regarded as the trigonometrical analogue of the decomposition of a rational function into partial fractions.

Example. Prove that

$$\begin{aligned}\frac{\sin(x-b_1)\sin(x-b_2)\ldots\sin(x-b_n)}{\sin(x-a_1)\sin(x-a_2)\ldots\sin(x-a_n)}&=\frac{\sin(a_1-b_1)\ldots\sin(a_1-b_n)}{\sin(a_1-a_2)\ldots\sin(a_1-a_n)}\cot(x-a_1)\\&+\frac{\sin(a_2-b_1)\ldots\sin(a_2-b_n)}{\sin(a_2-a_1)\ldots\sin(a_2-a_n)}\cot(x-a_2)\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+\cos(a_1+a_2+\ldots+a_n-b_1-b_2-\ldots-b_n).\end{aligned}$$

77. *Expansion in inverse factorials.*

Another mode of development of functions, which although investigated by Schlömilch as long ago as 1863 has hitherto not been much used*, is that of *expansion in inverse factorials.*

Let l be a line drawn parallel to the imaginary axis in the z-plane; and draw a circle of large radius, having its centre at the point where l cuts the real axis.

Consider a function $f(z)$, which has no singularities within the semi-circular area which is bounded by l and this circle and which lies on the positive side of l; let γ be the semi-circular arc which bounds this region. Suppose moreover that at all points of γ we have the inequality

$$|f(z)|<M$$

satisfied, where M is finite however large the radius of γ may be chosen.

Then if z be a point within this semi-circular region, we have

$$f(z)=\frac{1}{2\pi i}\left\{\int_l+\int_\gamma\right\}\frac{f(t)\,dt}{t-z}.$$

Now

$$\int_\gamma\frac{f(t)\,dt}{t-z}=\int_\gamma\frac{f(t)\,dt}{t}+\int_\gamma\frac{zf(t)\,dt}{t(t-z)}.$$

* References to some recent work are given by Kluyver, *Comptes Rendus*, CXXXIV. (1902), p. 587.

But

$$\left|\int_\gamma \frac{zf(t)\,dt}{t(t-z)}\right| < |z|\, M \int_\gamma \frac{dt}{|t|\,|t-z|},$$

which is infinitesimal when the radius of γ is infinitely great.

Thus

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t} - \frac{1}{2\pi i}\int_l \frac{f(t)\,dt}{t-z},$$

if we now suppose that the direction of integration along l is from $-i\infty$ to $+i\infty$.

Now if n be any positive integer and z be not equal to $0, -1, -2$, etc., we have the identity

$$\frac{1}{z-t} = \frac{1}{z} + \frac{t}{z(z+1)} + \frac{t(t+1)}{z(z+1)(z+2)} + \dots + \frac{t(t+1)\dots(t+n)}{z(z+1)\dots(z+n)(z-t)};$$

on substituting this in the second integral we have therefore

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z(z+1)} + \dots + \frac{a_{n+1}}{z(z+1)\dots(z+n)}$$
$$+ \frac{1}{2\pi i}\int_l \frac{f(t)\,t(t+1)\dots(t+n)}{z(z+1)\dots(z+n)(z-t)}\,dt,$$

where

$$a_0 = \frac{1}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t},$$

$$a_1 = \frac{1}{2\pi i}\int_l f(t)\,dt,$$

$$a_2 = \frac{1}{2\pi i}\int_l f(t)\,t\,dt,$$

....

$$a_{n+1} = \frac{1}{2\pi i}\int_l f(t)\,t(t+1)\dots(t+n-1)\,dt.$$

Now the product

$$\frac{t(t+1)\dots(t+n)}{z(z+1)\dots(z+n)}$$

can be written

$$\frac{t}{z}\prod_{r=1}^{n}\left\{\frac{1+\frac{t}{r}}{1+\frac{z}{r}}\right\}$$

and it diverges to zero or to infinity when z tends to ∞ according as the real part of $t-z$ is negative or positive, as can be seen by comparing it with the product

$$\prod_{r=1}^{n}\left(1+\frac{1}{r}\right)^{t-z},$$

which has the value $(n+1)^{t-z}$. But the real part of $t-z$ is, in the case under consideration, negative; and so the product

$$\frac{t(t+1)\dots(t+n)}{z(z+1)\dots(z+n)}$$

is infinitesimal when n is infinite.

Since $f(t)$ is finite along l, and $\int_l \frac{|dt|}{|z-t|}$ is finite, we see that

$$\int_l \frac{f(t)\,t(t+1)\dots(t+n)}{z(z+1)\dots(z+n)(z-t)}\,dt$$

is infinitesimal when n is infinite.

We can therefore expand $f(z)$ in the form

$$f(z)=a_0+\frac{a_1}{z}+\frac{a_2}{z(z+1)}+\frac{a_3}{z(z+1)(z+2)}+\dots,$$

the coefficients a being given by the above equation; and this expansion is valid for all values of z whose real part is greater than the real part of z at any of the singular points of $f(z)$, except for the points

$$z=0,\ -1,\ -2,\ \dots.$$

Example 1. Obtain the same result by using the equalities

$$\frac{1}{z(z+1)(z+2)\dots(z+n)}=\frac{1}{n!}\int_0^1 u^n(1-u)^{z-1}du,$$

$$\int_C \frac{f(t)\,dt}{z-t}=\int_C dt\int_0^1 du\,f(t)(1-u)^{z-t-1}.$$

Example 2. Obtain the expansion

$$\log\left(1+\frac{1}{z}\right)=\frac{1}{z}-\frac{a_1}{z(z+1)}+\frac{a_2}{z(z+1)(z+2)}\dots\dots,$$

where $$a_n=\int_0^1 t(1-t)(2-t)\dots(n-1-t)\,dt,$$

and discuss the region of its convergency. (Schlömilch.)

MISCELLANEOUS EXAMPLES.

1. Let $e^{-z^2}P_n$ denote the nth derivate of e^{-z^2}, so that

$$P_0=1,\quad P_1=-2z,\quad P_2=4z^2-2,\quad \text{etc.}$$

Shew that if $f(z)$ is an arbitrary function, then $f(z)$ can be expanded in the form

$$f(z)=a_0P_0+a_1P_1+a_2P_2+\dots,$$

where $$a_n=\frac{1}{2.4.6\dots2n\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-x^2}P_n(x)f(x)\,dx,$$

and find the region of convergence of this series. (Hermite.)

2. Obtain (from Darboux's formula or otherwise) the expansion

$$f(z)-f(a)=\frac{z-a}{1-r}\{f'(z)-rf'(a)\}$$
$$-\frac{(z-a)^2}{2!\,(1-r)^2}\{f''(z)-r^2f''(a)\}$$
$$+\ldots\ldots\ldots\ldots$$
$$+\frac{(-1)^{n-1}(z-a)^n}{n!\,(1-r)^n}\{f^{(n)}(z)-r^nf^{(n)}(a)\}$$
$$+\ldots\ldots\ldots\ldots;$$

find the remainder after n terms, and discuss the convergence of the series.

3. Shew that

$$f(x+h)-f(x)=\frac{h}{2}\{f'(x+h)+f'(x)\}$$
$$-\frac{1\,.\,3}{(2!)}\frac{h^2}{2^2}\{f''(x+h)-f''(x)\}+\ldots$$
$$+(-1)^{n-1}\frac{1\,.\,3\,.\,5\ldots(2n-1)}{(n!)}\frac{h^n}{2^n}\{f^n(x+h)+(-1)^nf^n(x)\}$$
$$+(-1)^nh^{n+1}\int_0^1\gamma_n(t)f^{n+1}(x+ht)\,dt,$$

where $$\gamma_n(x)=\frac{1}{n!\,(n-2)!}x^{n+\frac{1}{2}}(1-x)^{n+\frac{1}{2}}\frac{d^n}{dx^n}\{x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\}$$
$$=\frac{1}{\pi n!}\int_0^1(x-z)^nz^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}dz,$$

and shew that $\gamma_n(x)$ is the coefficient of $n!\,t^n$ in the expansion of $\{(1-tx)(1+t-tx)\}^{-\frac{1}{2}}$ in ascending powers of t.

4. By taking

$$\phi(x+1)=\frac{1}{n!}\left[\frac{d^n}{du^n}\left\{\frac{(1-r)\,e^{xu}}{1-re^{-u}}\right\}\right]_{u=0}$$

in Darboux's formula, shew that

$$f(x+h)-f(x)=-a_1h\left\{f'(x+h)-\frac{1}{r}f'(x)\right\}$$
$$-a_2\frac{h^2}{2!}\left\{f''(x+h)-\frac{1}{r}f''(x)\right\}+\ldots$$
$$-a_n\frac{h^n}{n!}\left\{f^n(x+h)-\frac{1}{r}f^n(x)\right\}$$
$$+(-1)^nh^{n+1}\int_0^1\phi_n(t)f^{n+1}(x+ht)\,dt,$$

where $$\frac{1-r}{1-re^{-u}}=1-a_1\frac{u}{1}+a_2\frac{u^2}{2!}-a_3\frac{u^3}{3!}+\ldots\ldots$$

5. Shew that

$$f(z)-f(a)=\frac{2B_1(2^2-1)(z-a)}{2!}\{f'(a)+f'(z)\}$$

$$-\frac{2B_2(2^4-1)(z-a)^3}{4!}\{f'''(a)+f'''(z)\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^{n-1}\frac{2B_n(2^{2n}-1)(z-a)^{2n-1}}{2n!}\{f^{(2n+1)}(a)+f^{(2n+1)}(z)\}$$

$$+\frac{(z-a)^{2n+1}}{2n!}\int_0^1\psi_{2n}(t)f^{(2n+1)}\{a+t(z-a)\}\,dt,$$

where $$\psi_n(t)=\frac{2}{n+1}\left[\frac{d^{n+1}}{du^{n+1}}\left(\frac{ue^{tu}}{e^u+1}\right)\right]_{u=0}.$$

6. Prove that

$$f(z_2)=f(z_1)+C_1(z_2-z_1)f'(z_2)+C_2(z_2-z_1)^2f''(z_1)+C_3(z_2-z_1)^3f'''(z_2)$$
$$+C_4(z_2-z_1)^4f^{\text{iv}}(z_1)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where C_n is the coefficient of z^n in the expansion of $\cot\left(\frac{\pi}{4}-\frac{z}{2}\right)$ in ascending powers of z.

(Trinity College Examination.)

7. If x_1 and x_2 are integers, and $\phi(z)$ is a function which is regular for all values of z (finite or infinite) of which the real part lies between x_1 and x_2, shew (by integrating

$$\int\frac{\phi(z)\,dz}{e^{2\pi iz}-1}$$

round a rectangle whose sides are parallel to the real and imaginary axes) that

$$\tfrac{1}{2}\phi(x_1)+\phi(x_1+1)+\phi(x_1+2)+\ldots+\phi(x_2-1)+\tfrac{1}{2}\phi(x_2)$$

$$=\int_0^{x_2}\phi(z)\,dz+\frac{1}{i}\int_0^\infty\frac{\phi(x_2+iy)-\phi(x_1+iy)-\phi(x_2-iy)+\phi(x_1-iy)}{e^{2\pi y}-1}dy.$$

Hence by applying the theorem

$$4n\int_0^\infty\frac{y^{2n-1}}{e^{2\pi y}-1}=B_{2n-1},$$

where B_1, B_3, ... are the Bernoullian numbers, shew that

$$\phi(1)+\phi(2)+\ldots+\phi(n)=C+\tfrac{1}{2}\phi(n)+\int^n\phi(z)\,dz+\sum_{n=1}^{\infty}\frac{(-1)^{r-1}B_{2r-1}}{2r!}\phi^{(2r-1)}(n),$$

(where C is a constant not involving n) provided that the last series converges.

8. Obtain the expansion

$$u=\frac{x}{2}+\sum_{n=2}^{n=\infty}(-1)^{n-1}\frac{1\cdot3\ldots(2n-3)}{n!}\frac{x^2}{2^n}$$

for one root of the equation $x=2u+u^2$, and shew that it converges so long as $|x|<1$.

9. If $S^{(m)}_{2n+1}$ denote the sum of all combinations of the numbers

$$1^2,\ 3^2,\ 5^2,\ \dots\ (2n-1)^2$$

taken m together, shew that

$$\frac{\cos z}{z} = \frac{1}{\sin z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)!} \left\{ \frac{2^{2(n+1)}}{2n+3} - S^{(1)}_{2(n+1)} \frac{2^{2n}}{2n+1} + \dots + S^{(n)}_{2(n+1)} \frac{2^2}{3} \right\} \sin^{2n+1} z.$$

10. If the function $f(z)$ is regular in the interior of that one of the ovals whose equation is $|\sin z| = C$ (where $C \leqslant 1$), which includes the origin, shew that $f(z)$ can, for all points z within this oval, be expanded in the form

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(2n)}(0) + S^{(1)}_{2n} f^{(2n-2)}(0) + \dots + S^{(n-1)}_{2n} f^{(2)}(0)}{2n\,!} \sin^{2n} z$$

$$+ \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0) + S^{(1)}_{2n+1} f^{(2n-1)}(0) + \dots + S^{(n)}_{2n+1} f'(0)}{(2n+1)\,!} \sin^{2n+1} z,$$

where $S^{(m)}_{2n}$ is the sum of all combinations of the numbers

$$2^2,\ 4^2,\ 6^2,\ \dots\ (2n-2)^2$$

taken m together, and $S^{(m)}_{2n+1}$ denotes the sum of all combinations of the numbers

$$1^2,\ 3^2,\ 5^2,\ \dots\ (2n-1)^2,$$

taken m together.

11. Shew that the two series

$$2z + \frac{2z^3}{3^2} + \frac{2z^5}{5^2} + \dots\dots$$

and

$$\frac{2z}{1-z^2} - \frac{2}{1\,.\,3^2}\left(\frac{2z}{1-z^2}\right)^2 + \frac{2\,.\,4}{3\,.\,5^2}\left(\frac{2z}{1-z^2}\right)^3 \dots$$

represent the same function in one part of the plane, and can be transformed into each other by Burmann's theorem.

12. If a function $f(z)$ is periodic, of period 2π, and is regular in the infinite strip of the plane, included between the two branches of the curve $|\sin z| = C$ (where $C > 1$), shew that at all points in the strip it can be expanded in an infinite series of the form

$$f(z) = A_0 + A_1 \sin z + \dots + A_n \sin^n z + \dots\dots$$
$$+ \cos z\,(B_1 + B_2 \sin z + \dots + B_n \sin^{n-1} z + \dots);$$

and find the coefficients A and B.

13. If ϕ and f be connected by the equation

$$\phi(x) + \lambda f(x) = 0,$$

of which one root is a, shew that

$$F(x) = F - \frac{\lambda}{1}\frac{1}{\phi'} fF' + \frac{\lambda^2}{2\,!}\frac{1}{\phi'^3}\begin{vmatrix} \phi' & f^2F' \\ \phi'' & (f^2F')' \end{vmatrix} \frac{1}{1}$$

$$- \frac{\lambda^3}{3\,!}\frac{1}{\phi'^6}\begin{vmatrix} \phi' & (\phi^2)' & (f^3F') \\ \phi'' & (\phi^2)'' & (f^3F')' \\ \phi''' & (\phi^3)''' & (f^3F')'' \end{vmatrix} \frac{1}{1\,.\,1\,.\,2} + \dots\dots,$$

where F, f, F', etc. denote

$$F(a),\ f(a),\ \frac{dF(a)}{da}\ \ldots\ldots$$

14. If a function $W(a, b, x)$ be defined by the series

$$W(a,\ b,\ x)=x+\frac{a-b}{2!}x^2+\frac{(a-b)(a-2b)}{3!}x^3+\ldots\ldots,$$

which converges so long as $$|x|<\frac{1}{|b|},$$

shew that $$\frac{d}{dx}W(a,\ b,\ x)=1+(a-b)\,W(a-b,\ b,\ x);$$

and shew that if $$y=W(a,\ b,\ x),$$

then $$x=W(b,\ a,\ y).$$

Examples of this function are

$$W(1,\ 0,\ x)=e^x-1,$$

$$W(0,\ 1,\ x)=\log(1+x),$$

$$W(a,\ 1,\ x)=\frac{(1+x)^a-1}{a}.$$

(Ježek.)

15. Prove that

$$\frac{1}{\sum\limits_{n=0}^{\infty} a_n x^n}=\frac{1}{a_0}+\sum_{1}^{\infty}\frac{(-1)^n x^n}{n!\,a_0^{\,n+1}}G_n,$$

where $$G_n=\begin{vmatrix} 2a_1 & a_0 & 0 & 0 & \ldots & 0 \\ 4a_2 & 3a_1 & 2a_0 & 0 & \ldots & 0 \\ 6a_3 & 5a_2 & 4a_1 & 3a_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (2n-2)\,a_{n-1} & \ldots & \ldots & \ldots & \ldots & (n-1)\,a_0 \\ na_n & (n-1)\,a_{n-1} & \ldots & \ldots & \ldots & a_1 \end{vmatrix},$$

and obtain a similar expression for

$$\left\{\sum_{n=0}^{\infty} a_n x^n\right\}^{\frac{1}{2}}.$$

(Mangeot.)

16. Shew that

$$\frac{1}{\sum\limits_{0}^{n} a_r x^r}=\sum_{0}^{\infty}\frac{1}{r+1}\frac{\partial S_{r+1}}{\partial a_1}x^r,$$

where S_r is the sum of the rth powers of the roots of the equation

$$\sum_{0}^{n} a_r x^r=0.$$

(Gambroli.)

17. If $f_n(z)$ denote the nth derivate of $f(z)$, and if $f_{-n}(z)$ denote that one of the nth integrals of $f(z)$ which has an n-ple zero at $z=0$, shew that

$$f(z+x)\,g(z+x)=\sum_{n=-\infty}^{\infty} f_n(z)\,g_{-n}(x);$$

and obtain Taylor's series from this result, by putting $g(z)=1$. (Guichard.)

18. Shew that, if x be not an integer, the series

$$\Sigma \frac{2x+m+n}{(x+m)^2 (x+n)^2},$$

in which m and n receive in every possible way unequal values, zero or integers lying between $+I$ and $-I$, vanishes when I increases indefinitely.

(Cambridge Mathematical Tripos, Part I, 1895.)

19. Sum the infinite series

$$\sum_{n=-q}^{n=p} \left(\frac{1}{(-)^n x - a - n} + \frac{1}{n} \right),$$

where the value $n=0$ is omitted, and p, q are positive integers to be increased without limit.

(Cambridge Mathematical Tripos, Part I, 1896.)

20. If $F(x) = e^{\int_0^x x\pi \cot(x\pi)\, dx}$, shew that

$$F(x) = e^x \frac{\prod_{n=1}^{n=\infty} \left\{ \left(1 - \frac{x}{n}\right)^n e^{x + \frac{1}{2}\frac{x^2}{n}} \right\}}{\prod_{n=1}^{n=\infty} \left\{ \left(1 + \frac{x}{n}\right)^n e^{-x + \frac{1}{2}\frac{x^2}{n}} \right\}},$$

and that the function thus defined satisfies the relations

$$F(-x) = \frac{1}{F(x)},$$

$$F(x) F(1-x) = 2 \sin x\pi.$$

Further, if

$$\psi(z) = z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \dots$$

$$= -\int_0^z \log(1-z)\, d(\log z),$$

shew that

$$F(x) = e^{\frac{1}{2}\pi i x^2 \frac{1}{2\pi i} \psi(1 - e^{-2\pi i x})}$$

when

$$|1 - e^{-2\pi i x}| < 1.$$

(Trinity College Examination.)

21. Shew that

$$\left[1 + \left(\frac{k}{x}\right)^n\right] \left[1 + \left(\frac{k}{2\pi - x}\right)^n\right] \left[1 + \left(\frac{k}{2\pi + x}\right)^n\right] \left[1 + \left(\frac{k}{4\pi - x}\right)^n\right] \left[1 + \left(\frac{k}{4\pi + x}\right)^n\right] \dots\dots$$

$$= \frac{\prod_1^{\frac{n}{2}} \{1 - 2e^{-a_g} \cos(x + \beta_g) + e^{-2a_g}\}^{\frac{1}{2}} \{1 - 2e^{-a_g} \cos(x - \beta_g) + e^{-2a_g}\}^{\frac{1}{2}}}{2^{\frac{n}{2}} (1 - \cos x)^{\frac{n}{2}} e^{-k \cos \frac{\pi}{n}}}$$

where

$$a_g = k \sin \frac{2g-1}{n} \pi,$$

$$\beta_g = k \cos \frac{2g-1}{n} \pi,$$

and $$0 < x < 2\pi.$$ (Mildner.)

22. If $|x| < 1$ and a is not a positive integer, shew that

$$\sum_{n=1}^{\infty} \frac{x^n}{n-a} = \frac{2\pi i x^a}{1-e^{2a\pi i}} + \frac{x}{1-e^{2a\pi i}} \int_C \frac{t^{a-1} - x^{a-1}}{t-x} dt,$$

where C is a contour in the t-plane enclosing the points 0, x. (Lerch.)

23. If $\phi_1(z)$, $\phi_2(z)$, ... are any polynomials in z, and if $F(z)$ be any function, and if $\psi_1(z)$, $\psi_2(z)$, ... be polynomials defined by the equations

$$\int_a^b F(x) \frac{\phi_1(z) - \phi_1(x)}{z-x} dx = \psi_1(z),$$

$$\int_a^b F(x) \phi_1(x) \frac{\phi_2(z) - \phi_2(x)}{z-x} dx = \psi_2(z),$$

$$\int_a^b F(x) \phi_1(x) \phi_2(x) \dots \phi_{m-1}(x) \frac{\phi_m(z) - \phi_m(x)}{z-x} dx = \psi_m(z),$$

shew that
$$\begin{aligned}\int_a^b \frac{F(x)\,dx}{z-x} = \frac{\psi_1(z)}{\phi_1(z)} &+ \frac{\psi_2(z)}{\phi_1(z)\phi_2(z)} \\ &+ \frac{\psi_3(z)}{\phi_1(z)\phi_2(z)\phi_3(z)} + \dots\dots \\ &+ \frac{\psi_m(z)}{\phi_1(z)\phi_2(z)\dots\phi_m(z)} \\ &+ \frac{1}{\phi_1(z)\phi_2(z)\dots\phi_m(z)} \int_a^b F(x)\phi_1(x)\phi_2(x)\dots\phi_m(x) \frac{dx}{z-x}.\end{aligned}$$

24. A system of functions $p_0(z)$, $p_1(z)$, $p_2(z)$, ... is defined by the equations

$$p_0(z) = 1, \qquad p_{n+1}(z) = (z^2 + a_n z + b_n) p_n(z),$$

where a_n and b_n are given functions of n, which for $n = \infty$ tend respectively to the limits 0 and -1.

Shew that the region of convergence of a series

$$\Sigma e_n p_n(z),$$

where e_1, e_2, ... are independent of z, is a Cassini's oval with the foci $+1$, -1.

Shew that every analytic function $f(z)$, which is regular in the interior of the oval, can for points in this region be expanded in a series

$$f(z) = \Sigma (c_n + z c_n') p_n(z),$$

where

$$c_n = \frac{1}{2\pi i} \int (a_n + z) q_n(z) f(z) dz,$$

$$c_n' = \frac{1}{2\pi i} \int q_n(z) f(z) dz,$$

the integrals being taken round the boundary of the region, and the functions $q_n(z)$ being defined by

$$q_0(z)=\frac{1}{z^2+a_0z+b_0},\quad q_{n+1}(z)=\frac{1}{z^2+a_{n+1}z+b_{n+1}}q_n(z).\qquad \text{(Pincherle.)}$$

25. If $P_n(x)$ be the coefficient of $\frac{z^n}{n!}$ in the expansion of

$$\frac{2hz}{e^{hz}-e^{-hz}}e^{xz}$$

in ascending powers of z, so that

$$P_0=1,\quad P_1=x,\quad P_2=\frac{3x^2-h^2}{6}\text{, etc.,}$$

shew that

(1) $P_n(x)$ is a homogeneous polynomial of degree n in x and h,

(2) $\frac{dP_n}{dx}=P_{n-1}\qquad (n\geqslant 1)$,

(3) $\int_{-h}^{h}P_n(x)\,dx=0\qquad (n\geqslant 1)$,

(4) If $y=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+\ldots$, where $a_0, a_1, a_2, \ldots$ are real constants, then the mean value of $\frac{d^ry}{dx^r}$ in the interval from $x=-h$ to $x=+h$ is a_r. (Léauté.)

26. If $P_n(x)$ be defined as in the preceding example, shew that

$$P_{2m}=(-1)^m 2\frac{h^{2m}}{\pi^{2m}}\left(\cos\frac{\pi x}{h}-\frac{1}{2^{2m}}\cos\frac{2\pi x}{h}+\frac{1}{3^{2m}}\cos\frac{3\pi x}{h}+\ldots\right),$$

$$P_{2m+1}=(-1)^m 2\frac{h^{2m+1}}{\pi^{2m+1}}\left(\sin\frac{\pi x}{h}-\frac{1}{2^{2m+1}}\sin\frac{2\pi x}{h}+\frac{1}{3^{2m+1}}\sin\frac{3\pi x}{h}+\ldots\right).\qquad \text{(Appell.)}$$

CHAPTER VII.

Fourier Series.

78. *Definition of Fourier series; nature of the region within which a Fourier series converges.*

Series of the type

$$a_0 + a_1 \cos z + a_2 \cos 2z + a_3 \cos 3z + \dots + b_1 \sin z + b_2 \sin 2z + b_3 \sin 3z + \dots,$$

where $a_0, a_1, a_2, a_3, b_1, b_2, b_3, \dots$ are independent of z, are of great importance in many analytical investigations. They are called *Fourier Series.*

We have already seen that the region within which a series of ascending powers of z converges is always a *circle*; and the region within which a series of ascending and descending powers of z converges is *the ring-shaped space between two circles*; we are therefore led by analogy to expect that series of the Fourier type will likewise converge within a region of some definite character.

To investigate this question, write $e^{iz} = \zeta$.

The series becomes

$$a_0 + \frac{a_1 - ib_1}{2}\zeta + \dots + \frac{a_r - ib_r}{2}\zeta^r + \dots + \frac{a_1 + ib_1}{2}\zeta^{-1} + \dots + \frac{a_r + ib_r}{2}\zeta^{-r} + \dots.$$

This is a Laurent series in ζ; it will therefore be convergent, if at all, within a ring-shaped space bounded by two circles in the ζ-plane; that is, it will be convergent for values of ζ satisfying an inequality of the type

$$a < |\zeta| < b,$$

where a and b are positive constants.

Now let

$$z = x + iy;$$

then

$$\zeta = e^{iz} = e^{-y+ix},$$

so
$$|\zeta| = e^{-y},$$

and therefore the inequality becomes

$$\log a < -y < \log b.$$

This inequality defines a belt of the z-plane, bounded by the two lines $y = -\log a$ and $y = -\log b$; hence *the region of convergence of a Fourier series is a belt of the z-plane, bounded by two lines parallel to the real axis.*

It may however happen that the Laurent series in ζ is divergent for all values of ζ, in which case the Fourier series is divergent for all values of z; or, (and this is the most important case for our purpose,) it may happen that $a = b$, so that the region of convergence of the Laurent series narrows down to the circumference of a single circle in the ζ-plane; in this case the region of convergence of the Fourier series narrows down to a single line parallel to the real axis in the plane of the variable z.

If now the coefficients $a_0, a_1, a_2, \ldots b_1, b_2, \ldots$ are all real, considerations of symmetry shew that if the Fourier series is divergent for a value $z = a + ib$, it will also be divergent for the value $z = a - ib$; so if in this case the region of convergence narrows down to a line, that line can only be the real axis in the z-plane.

Hence *a Fourier series with real coefficients may converge only for real values of z, and diverge for all complex values of z.*

An example of this class of expansions is afforded by the series

$$\sin z - \frac{1}{2}\sin 2z + \frac{1}{3}\sin 3z - \frac{1}{4}\sin 4z + \ldots.$$

Writing this in the form

$$\frac{1}{2i}\left(e^{iz} - \frac{1}{2}e^{2iz} + \frac{1}{3}e^{3iz}\ldots\right) - \frac{1}{2i}\left(e^{-iz} - \frac{1}{2}e^{-2iz} + \frac{1}{3}e^{-3iz}\ldots\right),$$

we see that it diverges when z is not purely real; when z is purely real and not an odd multiple of π, the sum of the series is

$$\frac{1}{2i}\log(1 + e^{iz}) - \frac{1}{2i}\log(1 + e^{-iz}),$$

or
$$\frac{1}{2i}\log e^{iz},$$

or
$$\frac{1}{2}z + k\pi,$$

where k is some integer, as yet undetermined.

Now when $z = 0$ the sum of the series is seen directly to be 0; when $z = \frac{\pi}{2}$, the sum of the series is $\tan^{-1} 1$, or $\frac{\pi}{4}$; when $z = -\frac{\pi}{2}$ the sum is

$-\tan^{-1} 1$, or $-\frac{\pi}{4}$. In this way we see that *when z lies between $-\pi$ and $+\pi$, the integer k is zero.*

But k is no longer zero when z is greater than π; for each term of the series is clearly unaffected if $z+2\pi$ be written for z: hence the sum of the series must be the same for $z+2\pi$ as for z; and hence when $\pi < z < 3\pi$, the sum of the series is $\frac{1}{2}z-\pi$; so that *when z lies between π and 3π, the integer k is -1.*

Proceeding in this way, we see that *the sum of the Fourier series is $\frac{1}{2}z+k\pi$, where k is an integer chosen so as to make $\frac{1}{2}z+k\pi$ lie between* $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. This is important as shewing that *the sum of a Fourier series is not necessarily a continuous analytic function.* It is clear however that the sum of a Fourier series can have discontinuities only in the case in which the region of convergence narrows down to the real axis; in the other case when the region of convergence is a belt of finite and infinite breadth, the Laurent series in ζ represents an analytic function, and therefore the Fourier series in z does also.

Example. Shew that the series

$$\cos z - \frac{1}{2^2}\cos 2z + \frac{1}{3^2}\cos 3z - \ldots$$

converges only for real values of z, and that when $-\pi < z < +\pi$ its sum is $\frac{\pi^2}{12}-\frac{1}{4}z^2$.

For when z is real, the series is absolutely and uniformly convergent, as is seen by comparing it with the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$.

When z is complex, we have (putting $z=x+iy$)

$$\frac{1}{n^2}\cos nz = \frac{1}{2n^2}\{e^{i(nx+niy)}+e^{i(-nx-niy)}\};$$

now either $\frac{e^{-ny}}{2n^2}$ or $\frac{e^{+ny}}{2n^2}$ is infinite for $n=\infty$, so the terms of the series are ultimately infinitely great and the series diverges.

To find the sum when z is real, it has been shewn that when $-\pi < z < \pi$ we have

$$\tfrac{1}{2}z = \sin z - \tfrac{1}{2}\sin 2z + \tfrac{1}{3}\sin 3z \ldots.$$

This series is uniformly convergent in the interval (though not at its extremes $-\pi$ and π) and so can be integrated.

Thus $$c-\tfrac{1}{4}z^2 = \cos z - \frac{1}{2^2}\cos 2z + \frac{1}{3^2}\cos 3z - \ldots,$$

where c is a constant.

To find c put $z=0$, which gives

$$c = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12},$$

whence the result.

79. *Values of the coefficients in terms of the sum of a Fourier series, when the series converges at all points in a belt of finite breadth in the z-plane.*

The connexion between the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ of a Fourier series, and the sum of the series, can be easily found in the case in which the series converges in a belt of finite breadth in the z-plane. For in this case, as we have seen, the sum of the series is an analytic function of z. Let it be denoted by $f(z)$, so that

$$f(z) = a_0 + a_1 \cos z + a_2 \cos 2z + \ldots + b_1 \sin z + b_2 \sin 2z + \ldots.$$

Writing $\zeta = e^{iz}$, the series becomes

$$f(z) = a_0 + \frac{a_1 - ib_1}{2}\zeta + \ldots + \frac{a_r - ib_r}{2}\zeta^r + \ldots + \frac{a_1 + ib_1}{2}\zeta^{-1} + \ldots + \frac{a_r + ib_r}{2}\zeta^{-r} + \ldots,$$

and by Laurent's theorem the coefficients in this expansion are given by the equations

$$2\pi i \frac{a_r - ib_r}{2} = \int_C f(z)\,\zeta^{-r-1} d\zeta,$$

$$2\pi i \frac{a_r + ib_r}{2} = \int_C f(z)\,\zeta^{r-1} d\zeta,$$

where C is any circle in the ζ-plane, surrounding the origin and contained within the ring-shaped region in which the expanded function is regular. Now if the quantities a_r and b_r are all real, we see as before by symmetry that the real axis must be contained in the region of convergence in the z-plane, and therefore the circle of radius unity must be contained in the region of convergence in the ζ-plane, since this circle corresponds to the real axis in the z-plane. We can therefore take C to be a circle of radius unity, with the point $\zeta = 0$ as centre.

Now writing $\zeta = e^{iz}$ in the integrals, we have

$$\pi(a_r - ib_r) = \int_0^{2\pi} f(z)\, e^{-riz} dz,$$

$$\pi(a_r + ib_r) = \int_0^{2\pi} f(z)\, e^{riz} dz,$$

and so

$$a_r = \frac{1}{\pi}\int_0^{2\pi} f(z) \cos rz\, dz \qquad (r > 0),$$

$$b_r = \frac{1}{\pi}\int_0^{2\pi} f(z) \sin rz\, dz,$$

and $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, dz.$$

These equations give the values of the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, of the Fourier series, in terms of the sum $f(z)$ of the series, in the case in which the series converges over a belt of finite breadth in the z-plane. We shall see in the next article that the same formulae hold good in the more extended case, in which the series converges only for real values of z.

Example. Shew that the function $\frac{\sin z}{1-2k\cos z+k^2}$ can, when $k<1$, be expanded in a Fourier series of sines of multiples of z, valid for all points z situated in a belt, of width $-2\log k$, parallel to the real axis in the z-plane.

For we have

$$\frac{\sin z}{1-2k\cos z+k^2} = \frac{1}{2ik}\left\{\frac{1}{1-ke^{iz}} - \frac{1}{1-ke^{-iz}}\right\},$$

and this can be expanded in the form $\sum_{n=1}^{\infty} \frac{1}{2ik} k^n (e^{niz} - e^{-niz})$, provided $k\,|e^{iz}|$ and $k\,|e^{-iz}|$ are less than unity. This can only happen when their product k^2 is less than 1, i.e. when

$$-1<k<1.$$

When this condition is satisfied, on putting $z=x+iy$, it is clear that we must have $|e^{ix-y}|<\frac{1}{k}$ and $>k$, i.e. we must have $-y$ lying between $\log\left(\frac{1}{k}\right)$ and $\log k$, i.e. z must be within a belt of width $-2\log k$, parallel to the real axis. When these conditions are satisfied the expansion is valid, and so

$$\frac{\sin z}{1-2k\cos z+k^2} = \sum_{n=1}^{\infty} k^{n-1}\sin nz.$$

80. *Fourier's Theorem.*

We have already said that the most interesting cases of Fourier's series are those to which the investigation of the last article cannot be applied, on account of the fact that the series converges only for real values of z. It is therefore necessary to undertake another investigation, in which the assumptions of the last article are no longer made. The result to which we shall be led is known as *Fourier's theorem*, and may be stated thus:

If $f(z)$ be a quantity which depends on a variable z, and which is finite and has only a limited number of maxima and minima and of finite discontinuities in the interval $0 < z < 2\pi$, then the sum of the series

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mz + b_m \sin mz),$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos mt\,dt,$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin mt\,dt,$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt,$$

represents $f(z)$, at every point in the interval $0 < z < 2\pi$ for which $f(z)$ is continuous; and at every point in the interval $0 < z < 2\pi$ for which $f(z)$ is discontinuous, the sum of the series is the arithmetic mean of the two values of $f(z)$ at the discontinuity.

The discussion of Fourier's theorem given below is a modification of what is known as *Cauchy's second proof*, which was originally published in 1827 in the second volume of his *Exercices de Mathématiques*, and is reprinted in his *Collected Works*, Second Series, Vol. VII., p. 393).

This proof, (which in its original form was in some respects imperfect,) seems to have been little used by the mathematicians of the nineteenth century, who in the discussion of Fourier's theorem almost universally followed the exposition of Dirichlet (which is also reproduced later in this chapter); the importance of Cauchy's proof was shewn by A. Harnack in 1888. It may be observed that the restrictions placed on $f(z)$—as to its having only a limited number of maxima and minima, etc.—are *sufficient* but not *necessary* for the validity of the expansion.

To establish the theorem, we write the first $2k+1$ terms of the expansion in the form

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \frac{1}{\pi}\sum_{m=1}^{m=k}\int_0^{2\pi} f(t)\cos m(z-t)\,dt,$$

or

$$\sum_{m=-k}^{m=k}\frac{1}{2\pi}\int_0^{2\pi} e^{mi(z-t)} f(t)\,dt,$$

or

$$U_k + V_k,$$

where

$$U_k = \sum_{m=-k}^{m=k}\frac{1}{2\pi}\int_0^{z} e^{mi(z-t)} f(t)\,dt,$$

$$V_k = \sum_{m=-k}^{m=k}\frac{1}{2\pi}\int_z^{2\pi} e^{mi(z-t)} f(t)\,dt.$$

We shall now investigate the behaviour of the quantity U_k when k, though finite, is a large number.

Let $\phi(\zeta)$ denote the quantity

$$\frac{1}{e^{2\pi\zeta}-1}\int_0^{z} e^{\zeta(z-t)} f(t)\,dt.$$

Then $\phi(\zeta)$ clearly has a definite value corresponding to every value of ζ, except the exceptional values $\zeta = 0, \pm i, \pm 2i$, for which $e^{2\pi\zeta} = 1$; moreover, it is easily seen that the quantity

$$\frac{\phi(\zeta + \delta\zeta) - \phi(\zeta)}{\delta\zeta}$$

tends to a definite limit when $\delta\zeta$ tends to zero, independently of the way in which $\delta\zeta$ tends to zero (still excepting the points $0, \pm i, \pm 2i \ldots$); hence $\phi(\zeta)$ is an analytic function of ζ, having poles at the points $0, \pm i, \pm 2i \ldots$; and the series U_k is clearly the sum of the residues of $\phi(\zeta)$ at those of its poles which are contained within a circle C_k in the ζ-plane, whose centre is at the origin and whose radius is $\left(k + \frac{1}{2}\right)$.

Hence $$U_k = \frac{1}{2\pi i}\int_{C_k} \phi(\zeta)\, d\zeta.$$

Write $$\zeta = \left(k + \frac{1}{2}\right) e^{i\theta}.$$

Thus $$U_k = \frac{1}{2\pi}\int_0^{2\pi} \zeta\phi(\zeta)\, d\theta.$$

Now we can write

$$U_k = \frac{1}{2\pi}\left\{\int_0^{\frac{\pi}{2}-k^{-\frac{1}{2}}} + \int_{\frac{\pi}{2}-k^{-\frac{1}{2}}}^{\frac{\pi}{2}+k^{-\frac{1}{2}}} + \int_{\frac{\pi}{2}+k^{-\frac{1}{2}}}^{\frac{3\pi}{2}-k^{-\frac{1}{2}}} + \int_{\frac{3\pi}{2}-k^{-\frac{1}{2}}}^{\frac{3\pi}{2}+k^{-\frac{1}{2}}} + \int_{\frac{3\pi}{2}+k^{-\frac{1}{2}}}^{2\pi}\right\} \zeta\phi(\zeta)\, d\theta$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \text{ say.}$$

At points in the range of integration of I_1 and I_5, the real part of ζ is positive and at least of order $k^{-\frac{1}{2}}$; and so for these integrals we have

$$\zeta\phi(\zeta) = \frac{\zeta}{e^{2\pi\zeta} - 1}\int_0^z e^{\zeta(z-t)} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2\pi\zeta}}\int_0^z \zeta e^{\zeta(z-t-2\pi)} f(t)\, dt.$$

In this expression, as k tends to infinity, $\frac{1}{1 - e^{-2\pi\zeta}}$ tends to the limit unity, and $\zeta e^{\zeta(z-t-2\pi)}$ tends to the limit zero: thus $\zeta\phi(\zeta)$ tends to the limit zero, since the range of integration from 0 to z is finite; and hence as I_1 and I_5 are the integrals of $\zeta\phi(\zeta)$ taken over finite ranges, we see that *I_1 and I_5 tend to zero as k tends to infinity.*

Considering next the integrals I_2 and I_4, we observe that the quantity $\frac{\zeta e^{\zeta(z-t)}}{e^{2\pi\zeta} - 1}$ is never infinite when $0 < t < z < 2\pi$, and so $\zeta\phi(\zeta)$ is never infinite;

and thus, since I_2 and I_4 are integrals of $\zeta\phi(\zeta)$ over ranges which become infinitesimal as k tends to infinity, it follows that *I_2 and I_4 tend to zero as k tends to infinity.*

Consider next the integral I_3, or

$$\frac{1}{2\pi}\int_{\frac{\pi}{2}+k^{-\frac{1}{2}}}^{\frac{3\pi}{2}-k^{-\frac{1}{2}}} \zeta\phi(\zeta)\,d\theta,$$

where $$\zeta\phi(\zeta) = \frac{1}{e^{2\pi\zeta}-1}\int_0^z \zeta e^{\zeta(z-t)} f(t)\,dt.$$

In the range of integration of I_3, the real part of ζ is negative and at least of order $k^{\frac{1}{2}}$. The factor $\dfrac{1}{e^{2\pi\zeta}-1}$ therefore tends to the value -1 as k tends to infinity; denote it by $-(1+\alpha_k)$, where α_k tends to zero as k tends to infinity.

Now $$\int_0^z \zeta e^{\zeta(z-t)} f(t)\,dt = J_1 + J_2,$$

where $$J_1 = \int_0^{z-\frac{1}{\log k}} \zeta e^{\zeta(z-t)} f(t)\,dt,$$

and $$J_2 = \int_{z-\frac{1}{\log k}}^{z} \zeta e^{\zeta(z-t)} f(t)\,dt.$$

Considering first J_1, we see that within its range of integration the quantity $\zeta(z-t)$ has its real part always negative and at least of order $\dfrac{k^{\frac{1}{2}}}{\log k}$, which tends to infinity with k; hence the quantity $\zeta e^{\zeta(z-t)}$ tends to zero as k tends to infinity; and therefore as the range of integration in J_1 is finite, we see that J_1 tends to zero as k tends to infinity.

Consider next J_2. Writing $v = \zeta(z-t)$, we have

$$J_2 = \int_0^{\frac{\zeta}{\log k}} e^v f\left(z - \frac{v}{\zeta}\right) dv,$$

and writing $e^v = w$, this becomes

$$J_2 = \int_1^{e^{\frac{\zeta}{\log k}}} f\left(z - \frac{\log w}{\zeta}\right) dw.$$

Now as k tends to infinity, $e^{\frac{\zeta}{\log k}}$ and $\dfrac{\log w}{\zeta}$ tend to the limit zero. Let $f(z-0)$ denote $f(z)$ if z is a point at which the function $f(z)$ is a continuous

function, and at those points at which $f(z)$ is a discontinuous function let $f(z-0)$ denote that one of the two values of $f(z)$ which is continuous with the value of f for values smaller than z. Then since there cannot be another discontinuity within an infinitesimal distance of z, we can write

$$f\left(z-\frac{\log w}{\zeta}\right)=f(z-0)+\eta,$$

where η tends to zero as k tends to infinity; and so

$$J_2=f(z-0)\int_1^{e^{\frac{\zeta}{\log k}}} dw+\int_1^{e^{\frac{\zeta}{\log k}}} \eta\, dw$$

$$=-f(z-0)+e^{\frac{\zeta}{\log k}} f(z-0)+\int_1^{e^{\frac{\zeta}{\log k}}} \eta\, dw,$$

or $$J_2=-f(z-0)+\epsilon,$$

where ϵ tends to zero as k tends to infinity.

Thus

$$\zeta\phi(\zeta)=-(1+\alpha_k)\{J_1-f(z-0)+\epsilon\},$$

where α_k, J_1, and ϵ, each tend to zero as k tends to infinity. We can write this $\zeta\phi(\zeta)=f(z-0)+\tau$, where τ tends to zero as k tends to infinity; and this is true throughout the range of integration of the integral I_3.

Thus

$$I_3=\frac{1}{2\pi}\int_{\frac{\pi}{2}+k^{-\frac{1}{2}}}^{\frac{3\pi}{2}-k^{-\frac{1}{2}}} \{f(z-0)+\tau\}\, d\theta,$$

or $I_3=\frac{1}{2}f(z-0)+\sigma$, where σ tends to zero as k tends to infinity.

Hence finally

$$U_k=\frac{1}{2}f(z-0)+\sigma+I_1+I_2+I_4+I_5,$$

where σ, I_1, I_2, I_4, I_5 each tend to zero as k tends to infinity; which can be written

$$U_k=\frac{1}{2}f(z-0)+u_k,$$

where u_k tends to zero as k tends to infinity.

Similarly we can shew that

$$V_k=\frac{1}{2}f(z+0)+v_k,$$

where v_k tends to zero as k tends to infinity, and where $f(z+0)$ denotes $f(z)$

if z is a value for which the function $f(z)$ is continuous, and denotes the value of $f(z)$ for values slightly greater than z if z is a value for which the function $f(z)$ is discontinuous. Hence the sum of the first $(2k+1)$ terms of the Fourier series is

$$\frac{1}{2}f(z-0)+\frac{1}{2}f(z+0)+u_k+v_k,$$

where u_k and v_k tend to zero as k tends to infinity; the sum to infinity of the series is therefore

$$\frac{1}{2}\{f(z-0)+f(z+0)\},$$

which establishes Fourier's theorem.

It must be observed that the sum of the series coincides with $f(z)$ *only for values of z between* 0 *and* 2π; outside these limits the sum $S(z)$ of the series can be found from the circumstance that $S(z+2n\pi)=S(z)$, (a result which is obvious, since all the terms are periodic); while $f(z)$ may of course have any values whatever when z is not included between the limits 0 and 2π.

Example. Take a function $f(z)$ such that

$$f(z)=\frac{\pi}{4} \text{ from } z=0 \text{ to } z=\pi,$$

and

$$f(z)=-\frac{\pi}{4} \text{ from } z=\pi \text{ to } z=2\pi.$$

The corresponding Fourier series is

$$a_0+\Sigma a_m \cos mz+\Sigma b_m \sin mz,$$

where

$$a_m=\frac{1}{\pi}\int_0^{2\pi} f(t)\cos mt\,dt,$$

$$b_m=\frac{1}{\pi}\int_0^{2\pi} f(t)\sin mt\,dt,$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt.$$

These integrals give

$$a_0=0,\quad a_m=\frac{1}{4}\int_0^{\pi}\cos mt\,dt-\frac{1}{4}\int_{\pi}^{2\pi}\cos mt\,dt=0,$$

$$b_m=\frac{1}{4}\int_0^{\pi}\sin mt\,dt-\frac{1}{4}\int_{\pi}^{2\pi}\sin mt\,dt=\frac{1}{4m}(1-\cos m\pi).$$

Therefore $b_m=0$ if m is even, and $b_m=\frac{1}{m}$ if m is odd; and so we have

$$f(z)=\frac{\sin z}{1}+\frac{\sin 3z}{3}+\frac{\sin 5z}{5}+\dots,$$

which is the required Fourier expansion.

This series can be summed by elementary methods in the following manner. We have

$$\sin z + \frac{\sin 3z}{3} + \frac{\sin 5z}{5} + \ldots = \frac{1}{2i}\left(e^{iz} + \frac{e^{3iz}}{3} + \ldots\right) - \frac{1}{2i}\left(e^{-iz} + \frac{e^{-3iz}}{3} + \ldots\right)$$

$$= \frac{1}{4i}\left(e^{iz} + \frac{e^{2iz}}{2} + \ldots\right) + \frac{1}{4i}\left(e^{iz} - \frac{e^{2iz}}{2} + \ldots\right) - \frac{1}{4i}\left(e^{-iz} + \frac{e^{-2iz}}{2} + \ldots\right) - \frac{1}{4i}\left(e^{-iz} - \frac{e^{-2iz}}{2} + \ldots\right)$$

$$= \frac{1}{4i}\log\frac{(1+e^{iz})(1-e^{-iz})}{(1-e^{iz})(1+e^{-iz})} = \frac{1}{4i}\log e^{iz - iz + i\pi} = \frac{\pi}{4} + \frac{r\pi}{2},$$

where r is an undetermined integer. It is clear from the above that r actually has the value zero when $0 < z < \pi$, and unity when $\pi < z < 2\pi$.

81. *The representation of a function by Fourier series for ranges other than* 0 *to* 2π.

Suppose now that the range of values of z, for which it is required to represent a function $f(z)$ by a Fourier series, is not the range from 0 to 2π, but from a to b, where a and b are any given real numbers. To extend Fourier's result to this case, we take a new variable z defined by the equation

$$z = a + \frac{b-a}{2\pi} z',$$

and write

$$f\left(a + \frac{b-a}{2\pi} z'\right) = F(z').$$

Then $F(z')$ is a function whose value is given for all values of its argument z' between 0 and 2π.

Therefore by the previous result we have

$$F(z') = \frac{1}{2\pi}\int_0^{2\pi} F(t')\,dt' + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_0^{2\pi} \cos m(z'-t')\,F(t')\,dt',$$

or writing

$$t = a + \frac{b-a}{2\pi} t',$$

we have

$$f(z) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{2}{b-a}\sum_{m=1}^{\infty}\int_a^b \cos\frac{2\pi m(z-t)}{(b-a)} f(t)\,dt.$$

This last result may be regarded as the *general form of Fourier's theorem*.

Example. To express the function $\dfrac{e^{mz} - e^{-mz}}{e^{m\pi} - e^{-m\pi}}$ as a Fourier series, valid when

$$-\pi < z < \pi.$$

Here $a = -\pi$, $b = \pi$.

The formula therefore becomes

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\cos n(z-t)f(t)\,dt.$$

Since in this case $f(t)=-f(-t)$, this reduces to

$$\begin{aligned} f(z) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nz\int_0^{\pi}\frac{e^{mt}-e^{-mt}}{e^{m\pi}-e^{-m\pi}}\sin nt\,dt \\ &= \sum_{n=1}^{\infty}\sin nz\int_0^{\pi}\frac{(e^{mt}-e^{-mt})(e^{int}-e^{-int})}{\pi i\,(e^{m\pi}-e^{-m\pi})}\,dt \\ &= \sum_{n=1}^{\infty}\frac{\sin nz}{\pi i\,(e^{m\pi}-e^{-m\pi})}\left\{\frac{e^{(m+in)\pi}-e^{-(m+in)\pi}}{m+in}-\frac{e^{(m-in)\pi}-e^{-(m-in)\pi}}{m-in}\right\} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n\sin nz}{\pi i}\left(\frac{1}{m+in}-\frac{1}{m-in}\right) \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2n}{\pi\,(m^2+n^2)}\sin nz, \end{aligned}$$

which is the required expansion.

82. *The Sine and Cosine Series.*

We proceed to derive two particular cases of Fourier's theorem which are of frequent occurrence.

Suppose that a function $f(z)$ is given for a range 0 to l of values of the variable z, and that we require a series which shall represent $f(z)$ for these values of z, and which shall have the value $f(-z)$ for values of z between 0 and $-l$.

To obtain a series of this character, we write in the preceding result $a=-l$, $b=l$, $f(-z)=f(z)$. Thus we have

$$f(z)=\frac{1}{2l}\int_{-l}^{l} f(t)\,dt+\frac{1}{l}\sum_{m=1}^{\infty}\int_{-l}^{l}\cos\frac{m\pi(z-t)}{l}f(t)\,dt,$$

or

$$f(z)=\frac{1}{l}\int_0^{l} f(t)\,dt+\frac{2}{l}\sum_{m=1}^{\infty}\cos\frac{m\pi z}{l}\int_0^{l}\cos\frac{m\pi t}{l}f(t)\,dt,$$

which is called the *Cosine Series.*

If on the other hand we require a series which shall represent $f(z)$ for values of z between 0 and l, and shall have the sum $-f(-z)$ for values of z between 0 and $-l$, we write in the general result $a=-l$, $b=l$, $f(-z)=-f(z)$, and thus obtain

$$f(z)=\frac{2}{l}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{l}\int_0^{l}\sin\frac{m\pi t}{l}f(t)\,dt,$$

which is called the *Sine Series.*

Example 1. Expand $\frac{\pi-z}{2}\sin z$ in a cosine series, valid when $0<z<\pi$.

When $0<z<\pi$, we have by the formula just obtained

$$\frac{\pi-z}{2}\sin z=\frac{1}{2\pi}\int_0^\pi(\pi-t)\sin t\,dt+\frac{1}{\pi}\sum_{n=1}^\infty\cos mz\int_0^\pi(\pi-t)\sin t\cos mt\,dt$$

$$=\frac{1}{\pi}\left[_0^\pi-\frac{\pi-t}{2}\cos t\right]-\frac{1}{2\pi}\int_0^\pi\cos t\,dt$$

$$+\frac{1}{2\pi}\sum_{m=1}^\infty\cos mz\left\{\int_0^\pi(\pi-t)\sin(m+1)t\,dt-\int_0^\pi(\pi-t)\sin(m-1)t\,dt\right\}$$

$$=\frac{1}{2}+\frac{1}{2\pi}\cos z\left[_0^\pi-\frac{(\pi-t)\cos 2t}{2}\right]+\frac{1}{2\pi}\sum_{m=2}^\infty\cos mz\left[_0^\pi-\frac{(\pi-t)\cos\overline{m+1}\,t}{m+1}+\frac{(\pi-t)\cos\overline{m-1}\,t}{m-1}\right]$$

$$=\frac{1}{2}+\frac{1}{4}\cos z+\frac{1}{2\pi}\sum_{m=2}^\infty\cos mz\left[\frac{\pi}{m+1}-\frac{\pi}{m-1}\right]$$

$$=\frac{1}{2}+\frac{1}{4}\cos z-\sum_{m=2}^\infty\frac{\cos mz}{(m-1)(m+1)}.$$

The required series is therefore

$$\frac{1}{2}+\frac{1}{4}\cos z-\frac{1}{1.3}\cos 2z-\frac{1}{2.4}\cos 3z-\frac{1}{3.5}\cos 4z-\ldots.$$

It will be observed that it is only for values of z between 0 and π that the sum of this series is proved to be $\frac{\pi-z}{2}\sin z$; thus for instance when z has a value between 0 and $-\pi$, the sum of the series is not $\frac{\pi-z}{2}\sin z$, but $-\frac{\pi+z}{2}\sin z$; when z has a value between π and 2π, the sum of the series happens to be again $\frac{\pi-z}{2}\sin z$, but this must be regarded as a mere coincidence arising from the special function considered, and not from the general theorem.

Example 2. To expand $\frac{\pi z(\pi-z)}{8}$ in a sine series, valid when $0<z<\pi$.

We have

$$\frac{\pi z(\pi-z)}{8}=\frac{2}{\pi}\sum_{m=1}^\infty\sin mz\int_0^\pi\frac{\pi t(\pi-t)}{8}\sin mt\,dt$$

$$=\sum_{m=1}^\infty\sin mz\int_0^\pi\frac{t(\pi-t)}{4}\sin mt\,dt.$$

But
$$\int_0^\pi\frac{t(\pi-t)}{4}\sin mt\,dt=\left[_0^\pi\frac{-t(\pi-t)}{4}\frac{\cos mt}{m}\right]+\frac{1}{4m}\int_0^\pi(\pi-2t)\cos mt\,dt$$

$$=\left[_0^\pi\frac{(\pi-2t)\sin mt}{4m^2}\right]+\frac{1}{2m^2}\int_0^\pi\sin mt\,dt$$

$$=\frac{1-(-1)^m}{2m^3}.$$

Therefore

$$\frac{\pi z(\pi-z)}{8}=\sin z+\frac{\sin 3z}{3^3}+\frac{\sin 5z}{5^3}+\ldots.$$

Here again the sum of the series is $\frac{\pi z(\pi - z)}{8}$ only when z lies between 0 and π. Thus when z lies between π and 2π, the sum of the series is $\frac{\pi(z-2\pi)(z-\pi)}{8}$. The sum of the series for values of z beyond the limits 0 and π can be found at once from the equations $S(z) = -S(-z)$ and $S(z+2\pi) = S(z)$, where $S(z)$ denotes the sum of the series.

Example 3. Prove that, when $0 < z < \pi$,

$$\frac{\pi(\pi-2z)(\pi^2+2\pi z-2z^2)}{96} = \cos z + \frac{\cos 3z}{3^4} + \frac{\cos 5z}{5^4} + \dots.$$

For when $0 < z < \pi$ we have

$$\begin{aligned}
\frac{\pi(\pi-2z)(\pi^2+2\pi z-2z^2)}{96} &= \frac{2}{\pi}\sum_{m=1}^{\infty} \cos mz \int_0^\pi \frac{\pi(\pi-2t)(\pi^2+2\pi t-2t^2)}{96} \cos mt\,dt \\
\text{(integrating by parts)} \quad &= \sum_{m=1}^{\infty} \cos mz \int_0^\pi \frac{\pi t - t^2}{4m} \sin mt\,dt \\
\text{(integrating by parts)} \quad &= \sum_{m=1}^{\infty} \cos mz \int_0^\pi \frac{\pi - 2t}{4m^2} \cos mt\,dt \\
\text{(integrating by parts)} \quad &= \sum_{m=1}^{\infty} \cos mz \int_0^\pi \frac{1}{2m^3} \sin mt\,dt \\
&= \sum_{m=1}^{\infty} \frac{1-(-1)^m}{2m^4} \cos mz \\
&= \cos z + \frac{\cos 3z}{3^4} + \frac{\cos 5z}{5^4} + \dots.
\end{aligned}$$

Example 4. Shew that for values of z between 0 and π, e^{sz} can be expanded in the cosine series

$$\frac{2s}{\pi}(e^{s\pi}-1)\left(\frac{1}{2s^2} + \frac{\cos 2z}{s^2+4} + \frac{\cos 4z}{s^2+16} + \dots\right) - \frac{2s}{\pi}(e^{s\pi}-1)\left(\frac{\cos z}{s^2+1} + \frac{\cos 3z}{s^2+9} + \dots\right),$$

and draw graphs of the function e^{sz} and of the sum of the series.

Example 5. Shew that for values of z between 0 and π, the function $\frac{\pi(\pi-2z)}{8}$ can be expanded in the cosine series

$$\cos z + \frac{\cos 3z}{3^2} + \frac{\cos 5z}{5^2} + \dots,$$

and draw graphs of the function $\frac{\pi(\pi-2z)}{8}$ and of the sum of the series.

83. *Alternative proof of Fourier's theorem.*

Another proof of Fourier's theorem, based on an entirely different set of ideas, is due to Dirichlet*.

* *Collected Works*, Vol. I. pp. 133—160.

Consider first the sum of a limited number of terms of the series

$$a_0+\sum_{m=1}^{\infty}[a_m\cos mz+b_m\sin mz],$$

where

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(t)\,tdt,$$

$$a_m=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos mt\,dt\quad(m=1,\ 2,\ 3,\ \ldots),$$

$$b_m=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin mt\,dt\quad(m=1,\ 2,\ 3,\ \ldots),$$

and where z is supposed to be a real variable.

Since

$$a_m\cos mz+b_m\sin mz=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos m(t-z)\,dt,$$

we have the sum to $(2m+1)$ terms of the series expressed by the formula

$$S_m=\frac{1}{\pi}\int_0^{2\pi}\{\tfrac{1}{2}+\cos(t-z)+\cos 2(t-z)+\ldots+\cos m(t-z)\}f(t)\,dt$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{\sin(2m+1)\frac{t-z}{2}}{\sin\frac{t-z}{2}}f(t)\,dt$$

$$=\frac{1}{\pi}\int_{-\frac{z}{2}}^{\pi-\frac{z}{2}}\frac{\sin(2m+1)\theta}{\sin\theta}f(z+2\theta)\,d\theta$$

$$=\frac{1}{\pi}\int_0^{\pi-\frac{z}{2}}\frac{\sin(2m+1)\theta}{\sin\theta}f(z+2\theta)\,d\theta$$

$$+\frac{1}{\pi}\int_0^{\frac{z}{2}}\frac{\sin(2m+1)\theta}{\sin\theta}f(z-2\theta)\,d\theta.$$

We have therefore to investigate the value to which integrals of this class tend as m tends to infinity. Consider in general the value to which

$$I=\int_0^h\frac{\sin kz}{\sin z}\phi(z)\,dz$$

tends when k, supposed to be an odd integer, increases without limit.

First suppose $0<h<\frac{\pi}{2}$, and suppose that, for values of z within this range, $\phi(z)$ is continuous and positive, and that $\phi(z)$ continually decreases as z increases.

Let $\frac{r\pi}{k}$ be the greatest multiple of $\frac{\pi}{k}$ in h, so $\frac{r\pi}{k}<h<\overline{r+1}\,\frac{\pi}{k}$.

Then

$$I = \int_0^{\frac{\pi}{k}} + \int_{\frac{\pi}{k}}^{\frac{2\pi}{k}} + \ldots + \int_{\frac{n\pi}{k}}^{\frac{(n+1)\pi}{k}} + \ldots + \int_{\frac{(r-1)\pi}{k}}^{\frac{r\pi}{k}} + \int_{\frac{r\pi}{k}}^{h} \frac{\sin kz}{\sin z} \phi(z)\, dz.$$

Now write

$$u_n = (-1)^n \int_{\frac{n\pi}{k}}^{\frac{(n+1)\pi}{k}} \frac{\sin kz}{\sin z} \phi(z)\, dz,$$

so

$$u_n = \int_0^{\frac{\pi}{k}} \frac{\sin ky}{\sin\left(\frac{n\pi}{k} + y\right)} \phi\left(\frac{n\pi}{k} + y\right) dy, \quad \text{where } y = z - \frac{n\pi}{k}.$$

The integrand in this last integral is clearly positive throughout the range of integration, and u_n is therefore positive. Moreover, under the suppositions already stated, the quantity

$$\frac{\phi\left(\frac{n\pi}{k} + y\right)}{\sin\left(\frac{n\pi}{k} + y\right)}$$

decreases as n increases, and it therefore follows that u_n decreases as n increases.

Also the well-known theorem of Mean Value shews that u_n can be represented in the form

$$u_n = \rho_n \nu_n,$$

where

$$\nu_n = \int_0^{\frac{\pi}{k}} \frac{\sin ky}{\sin\left(\frac{n\pi}{k} + y\right)} dy,$$

and

$$\rho_n = \phi\left(\frac{n\pi}{k} + \theta\right),$$

θ being some quantity between 0 and $\frac{\pi}{k}$. Clearly ν_n is positive, and decreases as n increases.

Now we can write

$$I = J + \int_{\frac{r\pi}{k}}^{h} \frac{\sin kz}{\sin z} \phi(z)\, dz,$$

where

$$J = u_0 - u_1 + u_2 - u_3 + \ldots + (-1)^{r-1} u_{r-1}.$$

Since u_n is always positive, and decreases as n increases, we have

$$J < u_0 - u_1 + u_2 - \ldots + u_{2m},$$

where m is any number less than $\frac{r-1}{2}$.

This gives

$$\begin{aligned} J &< \nu_0\rho_0 - \nu_1\rho_1 + \nu_2\rho_2 - \ldots + \nu_{2m}\rho_{2m} \\ &< \nu_0\rho_0 - (\nu_1 - \nu_2)\rho_2 - (\nu_3 - \nu_4)\rho_4 - \ldots - (\nu_{2m-1} - \nu_{2m})\rho_{2m} \\ &\qquad - \nu_1(\rho_1 - \rho_2) - \nu_3(\rho_3 - \rho_4) - \ldots - \nu_{2m-1}(\rho_{2m-1} - \rho_{2m}). \end{aligned}$$

As ρ_n decreases with increase of n, the terms in the last line are negative, and can be removed without affecting the inequality.

Thus

$$J < \nu_0\rho_0 - (\nu_1 - \nu_2)\rho_2 - (\nu_3 - \nu_4)\rho_4 - \dots - (\nu_{2m-1} - \nu_{2m})\rho_{2m}$$
$$< \nu_0\rho_0 - (\nu_1 - \nu_2)\rho_{2m} - (\nu_3 - \nu_4)\rho_{2m} - \dots - (\nu_{2m-1} - \nu_{2m})\rho_{2m}$$
$$- (\nu_1 - \nu_2)(\rho_2 - \rho_{2m}) - (\nu_3 - \nu_4)(\rho_4 - \rho_{2m}) - \dots - (\nu_{2m-3} - \nu_{2m-2})(\rho_{2m-2} - \rho_{2m}).$$

The terms in the last line are again negative and can be removed. Thus

$$J < \nu_0(\rho_0 - \rho_{2m}) + (\nu_0 - \nu_1 + \nu_2 - \dots + \nu_{2m})\rho_{2m}.$$

We also have clearly

$$J > u_0 - u_1 + u_2 - \dots - u_{2m-1},$$

or

$$J > \nu_0\rho_0 - \nu_1\rho_1 + \nu_2\rho_2 - \dots - \nu_{2m+1}\rho_{2m+1},$$

which in the same way gives

$$J > \rho_{2m}(\nu_0 - \nu_1 + \nu_2 - \dots - \nu_{2m+1}).$$

Thus J is intermediate in value between the quantities

$$\nu_0(\rho_0 - \rho_{2m}) + (\nu_0 - \nu_1 + \nu_2 - \dots + \nu_{2m})\rho_{2m}$$

and

$$\rho_{2m}(\nu_0 - \nu_1 + \nu_2 - \dots - \nu_{2m+1}).$$

Now let k become infinitely great, and let the quantity m likewise become infinitely great, but in such a way that $\frac{m}{k}$ tends to the limit zero. Then the quantities ρ_0 and ρ_{2m} tend to the limit $\phi(0)$; and the quantity

$$\nu_0 - \nu_1 + \nu_2 - \dots + \nu_{2m}$$

or

$$\int_0^{\frac{(2m+1)\pi}{k}} \frac{\sin ky}{\sin y}\, dy$$

or

$$\int_0^{(2m+1)\pi} \frac{\sin t}{k \sin \frac{t}{k}}\, dt$$

can, since k is infinitely large compared with m, be replaced by

$$\int_0^{(2m+1)\pi} \frac{\sin t}{t}\, dt;$$

and this, when m becomes infinitely great, tends to the limit $\frac{\pi}{2}$.

We see therefore that J is intermediate in value between two quantities, each of which tends to the same limit, namely $\frac{\pi}{2}\phi(0)$. J therefore tends to the limit $\frac{\pi}{2}\phi(0)$; and therefore I, which differs from J only by a vanishing integral, likewise tends to the limit $\frac{\pi}{2}\phi(0)$ as k becomes infinitely great. This result may be called *Dirichlet's lemma*. To complete the lemma, however, it will be necessary to shew that it is still true when a number of the restrictions imposed on $\phi(z)$ are removed.

(1) It was assumed that $\phi(z)$ was positive and steadily decreasing throughout the range.

(*a*) Suppose that $\phi(z)$ is constant. This does not invalidate any of the preceding proof, so the theorem still holds if $\phi(z)$ is constant.

(β) Suppose that $\phi(z)$ is negative, or partly positive and partly negative, but still steadily decreasing; then choose a constant c so that $c+\phi(z)$ is positive through the range; then the theorem applies both to c and to $c+\phi(z)$ and therefore on subtraction to $\phi(z)$ alone.

(γ) Suppose that $\phi(z)$ increases steadily throughout the range. Then the theorem is true for $\{-\phi(z)\}$ and therefore for $\phi(z)$.

Therefore the theorem is still true if $\phi(z)$ is *finite, continuous*, and *steadily increases or decreases* throughout the range.

(2) Instead of taking the integral between 0 and h, take it between g and h, where $0 < g < h \leqslant \frac{\pi}{2}$. We assume that the value of $\phi(z)$ is only known for values of z from g to h.

Take a new function $\phi_1(z)$, defined as being equal to $\phi(g)$, a constant, for values of z from 0 to g, and equal to $\phi(z)$ for values of z from g to h. Then the theorem holds for $\phi_1(z)$.

Also

$$\operatorname*{Limit}_{k=\infty} \int_0^g \frac{\sin kz}{\sin z} \phi_1(z)\, dz = \frac{\pi}{2}\phi_1(0) = \frac{\pi}{2}\phi(g),$$

and

$$\operatorname*{Limit}_{k=\infty} \int_0^h \frac{\sin kz}{\sin z} \phi_1(z)\, dz = \frac{\pi}{2}\phi_1(0) = \frac{\pi}{2}\phi(g).$$

Therefore

$$\operatorname*{Limit}_{k=\infty} \int_g^h \frac{\sin kz}{\sin z} \phi(z)\, dz = 0,$$

by subtraction.

(3) Now assume there are a limited number n of maxima and minima within the range 0 to h.

Let them be at the values $a_1, a_2, \ldots a_n$, of z. Then

$$\int_0^h = \int_0^{a_1} + \int_{a_1}^{a_2} + \ldots + \int_{a_n}^h \frac{\sin kz}{\sin z} \phi(z)\, dz.$$

On applying the theorem to each of these integrals in succession, it is clear that the theorem holds for the whole integral.

Therefore the theorem is still true if $\phi(z)$ is finite, continuous, and has not more than a limited number of maxima and minima within the range.

It must be noted that these conditions still exclude such functions as e.g. $(z-c)\sin\frac{1}{z-c}$ where $0 < c < h$.

(4) We shall now no longer restrict h to be less than $\frac{\pi}{2}$. Take $0 < h \leqslant \pi$.

Then (a) let $\frac{\pi}{2} < h < \pi$.

Write $h = \pi - h'$, where $0 < h' < \frac{\pi}{2}$. Then

$$\operatorname*{Limit}_{k=\infty} I = \int_0^{\frac{\pi}{2}} \frac{\sin kz}{\sin z} \phi(z)\, dz + \int_{\frac{\pi}{2}}^{\pi-h'} \frac{\sin kz}{\sin z} \phi(z)\, dz.$$

Writing $z=\pi-\zeta$ in the latter integral, we have

$$\operatorname*{Limit}_{k=\infty} I=\int_0^{\frac{\pi}{2}}\frac{\sin kz}{\sin z}\phi(z)\,dz+\int_{h'}^{\frac{\pi}{2}}\frac{\sin k\zeta}{\sin\zeta}\phi(\pi-\zeta)\,d\zeta.$$

Since $\phi(\pi-\zeta)$ satisfies the conditions stated, we see that when $h'>0$ the second integral is zero.

Therefore $$\operatorname*{Limit}_{k=\infty} I=\frac{\pi}{2}\phi(0).$$

(β) Let $h=\pi$. Then all the above reasoning applies, except that now $h'=0$, so

$$\operatorname*{Limit}_{k=\infty} I=\frac{\pi}{2}\phi(0)+\frac{\pi}{2}\phi(\pi),$$

which, in order to guard against uncertainty in the case in which the function ϕ is discontinuous at 0 and π, is often written

$$I=\frac{\pi}{2}\{\phi(+\epsilon)+\phi(\pi-\epsilon)\},$$

where ϵ is a vanishing positive quantity.

(5) Next, suppose that the function $\phi(z)$ within the range has a finite number of discontinuities, in the form of abrupt but finite changes of value. Divide the range into various portions, so that each of them ends at one discontinuity and begins at the next, and divide each of these into others each beginning and ending at a point of stationary value. The above theorems apply to each of the portions, and therefore each integral is zero except the first, which is equal to $\frac{\pi}{2}\phi(\epsilon)$, and possibly the last, which when $h=\pi$ has the value $\frac{\pi}{2}\phi(\pi-\epsilon)$.

(6) Finally, consider a function $\phi(z)$ which becomes infinite for $z=c$, but in such a way that the value of $\int^z \phi(z)\,dz$ tends to a definite limit as z approaches c from either lower or greater values.

Then

$$I=\int_0^{c-\epsilon}+\int_{c-\epsilon}^{c}+\int_c^{c+\epsilon}+\int_{c+\epsilon}^{h}\frac{\sin kz}{\sin z}\phi(z)\,dz,$$

where ϵ is a small positive quantity.

In the second integral, a quantity ζ can be chosen intermediate between c and $c-\epsilon$, such that the integral is equal to $\frac{\sin k\zeta}{\sin\zeta}\int_{c-\epsilon}^{c}\phi(z)\,dz$; on taking ϵ small this vanishes; and similarly the third integral is zero.

On making k infinitely large, the fourth integral tends to zero. Therefore the theorem holds in this case also.

(7) Thus we have, summarising the results obtained, the theorem that *the limit when k tends to infinity of $\int_0^h \frac{\sin kz}{\sin z}\phi(z)\,dz$ is $\frac{\pi}{2}\phi(\epsilon)$ if $0<h<\pi$, and is*

$$\frac{\pi}{2}\{\phi(+\epsilon)+\phi(\pi-\epsilon)\}$$

if $h=\pi$; where ϵ is a vanishing positive quantity; provided that $\phi(z)$ is everywhere finite, and has only a limited number of finite discontinuities and maxima and minima between the values 0 *and* h *of the variable* z; *and this is still true if $\phi(z)$ has a limited number of singularities of specified type, namely such that* $\int \phi(z)\,dz$ *is finite.*

This result may be called *Dirichlet's lemma*, the conditions just stated being referred to as *Dirichlet's conditions.*

We can now return to the expansion which was found to represent the sum of the first $(2m+1)$ terms of the Fourier series.

We had
$$S_m = I_1 + I_2,$$
where
$$I_1 = \frac{1}{\pi}\int_0^{\pi-\frac{z}{2}} \frac{\sin(2m+1)\theta}{\sin\theta} f(z+2\theta)\,d\theta,$$
$$I_2 = \frac{1}{\pi}\int_0^{\frac{z}{2}} \frac{\sin(2m+1)}{\sin\theta} f(z-2\theta)\,d\theta.$$

If $0<z<2\pi$, and $f(z)$ satisfies Dirichlet's conditions, we have by Dirichlet's lemma
$$\operatorname*{Limit}_{m=\infty} I_1 = \tfrac{1}{2} f(z+\epsilon),$$
and
$$\operatorname*{Limit}_{m=\infty} I_2 = \tfrac{1}{2} f(z-\epsilon),$$
and so
$$\operatorname*{Limit}_{m=\infty} S_m = \tfrac{1}{2}\{f(z+\epsilon)+f(z-\epsilon)\}.$$

If $z=0$, we have
$$\operatorname*{Limit}_{m=\infty} I_1 = \tfrac{1}{2}\{f(\epsilon)+f(2\pi-\epsilon)\}, \qquad \operatorname*{Limit}_{m=\infty} I_2 = 0,$$
and so
$$\operatorname*{Limit}_{m=\infty} S_m = \tfrac{1}{2}\{f(\epsilon)+f(2\pi-\epsilon)\}.$$

If $z=2\pi$, we have
$$\operatorname*{Limit}_{m=\infty} I_1 = 0, \qquad \operatorname*{Limit}_{m=\infty} I_2 = \tfrac{1}{2}\{f(\epsilon)+f(2\pi-\epsilon)\},$$
and so
$$\operatorname*{Limit}_{m=\infty} S_m = \tfrac{1}{2}\{f(\epsilon)+f(2\pi-\epsilon)\}.$$

Thus we finally arrive at Fourier's theorem, namely that *the sum to infinity of the series*
$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mz + b_m \sin mz)$$
is $f(z)$ at points z for which f is continuous, and is the arithmetic mean of the two values of $f(z)$ at points z for which f is discontinuous: it being assumed that $f(z)$ satisfies Dirichlet's conditions.

Example. Prove that in the limit when n becomes infinitely great
$$\int_0^{\infty} \frac{\sin(2n+1)\theta}{\sin\theta} e^{-a\theta} d\theta = \tfrac{1}{2}\pi \coth \tfrac{1}{2}a\pi,$$
a being a real positive constant.

(Cambridge Mathematical Tripos, Part II., 1894.)

84. *Nature of the convergence of a Fourier series.*

The proofs of Fourier's theorem which have been given establish the result only for the case in which the sequence of the terms in the series

$$\Sigma (a_m \cos mz + b_m \sin mz)$$

is that in which m takes the orderly succession of values 1, 2, 3, 4, ...

The question now arises whether the order of succession of the terms can be deranged without affecting the value of the sum of the series; in other words, we have proved that the expansion of a function by Fourier's theorem is a convergent series: we want to find whether it is *absolutely convergent*, or only *semi-convergent*. The question has also to be considered whether the series is *uniformly convergent* or *non-uniformly convergent* in the neighbourhood of a given value of z.

We shall first shew by considering special cases that there is no general answer to these questions.

Consider the series

$$\sin z - \tfrac{1}{2} \sin 2z + \tfrac{1}{3} \sin 3z - \dots,$$

which represents $\frac{1}{2} z$ when $0 < z < \pi$, and $\frac{1}{2} z - \pi$ when $\pi < z < 2\pi$; this series is semi-convergent for all real values of z, since $\sin nz$ is finite for all values of n when z is real, and so the modulus of the general term bears a finite ratio to the general term of the divergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots.$$

In this series, therefore, the value of the sum will be modified if the order of succession of the terms is changed.

Moreover, we can shew that the series is non-uniformly convergent at its discontinuity π. For the sum of the first n terms is

$$\sin z - \frac{\sin 2z}{2} + \dots + \frac{(-1)^{n-1} \sin nz}{n},$$

or

$$\int_0^z (\cos t - \cos 2t + \dots + (-1)^{n-1} \cos nt)\, dt,$$

or

$$\int_0^z \left[\frac{1}{2} + \frac{(-1)^{n-1}}{2} \frac{\cos (n+1)t + \cos nt}{1 + \cos t}\right] dt.$$

The term $\int_0^z \frac{1}{2} dz$ represents the sum of the whole series; so the remainder after n terms, when $-\pi < z < \pi$, is

$$R_n = (-1)^{n-1} \int_0^z \frac{\cos\left(n + \frac{1}{2}\right) t}{2 \cos \frac{t}{2}} dt.$$

Writing $z = \pi - \eta$, $t = \pi - u$, this can be written

$$R_n = -\int_\eta^\pi \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\,du.$$

Write $\left(n+\frac{1}{2}\right)u = v$. The equation becomes

$$R_n = -\int_{(n+\frac{1}{2})\eta}^{(n+\frac{1}{2})\pi} \frac{\sin v}{(2n+1)\sin\frac{v}{2n+1}}\,dv.$$

However great n may be taken, if η be taken so small that $\left(n+\frac{1}{2}\right)\eta$ is infinitesimal, this integral tends to $-\int_0^\infty \frac{\sin v\,dv}{v}$ or $-\frac{\pi}{2}$, and so is not infinitesimal. It follows that the series is non-uniformly convergent in the vicinity of $z = \pi$.

Consider next the series

$$\cos z + \frac{1}{3^2}\cos 3z + \frac{1}{5^2}\cos 5z + \dots,$$

which represents $\frac{\pi(\pi - 2z)}{8}$ when $0 < z < \pi$, and $\frac{\pi(2z - 3\pi)}{8}$ when $\pi < z < 2\pi$. This series is absolutely convergent for all real values of z, since the moduli of its terms are less than the corresponding terms of the convergent series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

In this series therefore the order of succession of the terms can be changed in any way, without altering the value of the sum of the series; and since the comparison series is independent of z, the series is also *uniformly* convergent for all real values of z.

Returning now to the general Fourier series, we can discover the nature of the convergence by a consideration of the coefficients in the series, which can be made in the following way.

We have shewn that if

$$f(z) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mz + b_m \sin mz),$$

then

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos mt\,dt.$$

Suppose that (as in most of the examples we have discussed) the range $0 < z < 2\pi$ can be divided into other ranges, say $0 < z < k_1$, $k_1 < z < k_2$, ..., $k_n < z < 2\pi$, which are such that in each of these smaller ranges $f(z)$ is an analytic function of z, regular in the range. ($f(z)$ will not necessarily be the *same* analytic function in the different ranges.) Thus if $f(z)$ has the value z for $0 < z < \pi$, and has the value $-z$ for $\pi < z < 2\pi$, we should have $n = 1$ and $k_1 = \pi$. Then

$$a_m = \frac{1}{\pi}\int_0^{k_1} f(t)\cos mt\,dt + \frac{1}{\pi}\int_{k_1}^{k_2} f(t)\cos mt\,dt + \ldots + \frac{1}{\pi}\int_{k_n}^{2\pi} f(t)\cos mt\,dt.$$

Each of these integrals can then be integrated by parts; we thus obtain

$$a_m = \left[\frac{1}{\pi} f(t)\,\frac{\sin mt}{m}\right]_0^{k_1} + \left[\frac{1}{\pi} f(t)\,\frac{\sin mt}{m}\right]_{k_1}^{k_2} + \ldots$$

$$-\frac{1}{\pi m}\int_0^{k_1} f'(t)\sin mt\,dt - \frac{1}{\pi m}\int_{k_1}^{k_2} f'(t)\sin mt\,dt - \ldots,$$

or

$$a_m = \frac{A}{m} - \frac{b_m'}{m},$$

where

$$A = \frac{1}{\pi}[\sin mk_1\{f(k_1-0) - f(k_1+0)\} + \sin mk_2\{f(k_2-0) - f(k_2+0)\} + \ldots],$$

and where b_m' is the coefficient of $\sin mz$ in the Fourier expansion of $f'(z)$—an expansion which will exist, since $f'(z)$ is a function of the same character as $f(z)$, though the terms of this expansion will not always be the derivates of the corresponding terms of the Fourier series for $f(z)$.

Similarly

$$b_m = \frac{B}{m} + \frac{a_m'}{m},$$

where

$$B = -\frac{1}{\pi}[-f(+0) + \cos mk_1\{f(k_1-0) - f(k_1+0)\} + \cos mk_2\{f(k_2-0)$$
$$- f(k_2+0)\} + \ldots + f(2\pi - 0)],$$

and where a_m' is the coefficient of $\cos mz$ in the Fourier expansion of $f'(z)$.

In the same way we have

$$a_m' = \frac{A'}{m} - \frac{b_m''}{m},$$

where

$$A' = \frac{1}{\pi}[\sin mk_1\{f'(k_1-0) - f'(k_1+0)\} + \sin mk_2\{f'(k_2-0) - f'(k_2+0)\} + \ldots],$$

and

$$b_m' = \frac{B'}{m} + \frac{a_m''}{m},$$

where

$$B' = -\frac{1}{\pi}[-f'(+0) + \cos mk_1\{f'(k_1-0) - f'(k_1+0)\} + \ldots + f'(2\pi - 0)],$$

a_m'' and b_m'' being the coefficients of $\cos mz$ and $\sin mz$ respectively in the Fourier expansion of $f''(z)$.

Thus

$$a_m = \frac{A}{m} - \frac{B'}{m^2} - \frac{a_m''}{m^2},$$

$$b_m = \frac{B}{m} + \frac{A'}{m^2} - \frac{b_m''}{m^2}.$$

The conditions for the *absolute* convergence of the Fourier expansion of $f(z)$ are therefore expressed by the equations

$$A = 0, \quad B = 0;$$

for if these equations are satisfied, we have

$$a_m = -\frac{B' + a_m''}{m^2} \text{ and } b_m = \frac{A' - b_m''}{m^2},$$

and the terms of the Fourier series are comparable with those of the convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$

Now in order that we may have $A = 0$, $B = 0$, for all values of m we must have

$$f(k_1 - 0) = f(k_1 + 0),$$
$$f(k_2 - 0) = f(k_2 + 0),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f(k_n - 0) = f(k_n + 0),$$
$$f(2\pi - 0) = f(0).$$

That is to say, *if a Fourier series is absolutely convergent for all real values of z, the function represented by the series has no discontinuities, and has the same value at $z = 0$ as at $z = 2\pi$.*

If these conditions are satisfied the Fourier series is not only *absolutely*, but is also *uniformly* convergent. For its coefficients a_m and b_m are in this case of the order $\frac{1}{m^2}$, and so the series of constants

$$|a_0| + |a_1| + |b_1| + |a_2| + |b_2| + \ldots$$

converges; but the moduli of the terms of the Fourier series are less than the corresponding terms of this series, and consequently the Fourier series is uniformly convergent for all real values of z.

Example 1. Shew that in general, when the Fourier series converges only for real values of z, the quantities a_m and b_m can be expanded in infinite series of the form

$$\frac{c_1}{m}+\frac{c_2}{m^2}+\frac{c_3}{m^3}+\frac{c_4}{m^4}+\dots,$$

of which the terms

$$\frac{A}{m}-\frac{B'}{m^2} \text{ and } \frac{B}{m}+\frac{A'}{m^2}$$

found above are the initial terms; but that when the Fourier series converges within a belt of finite breadth in the z-plane, all the coefficients c_1, c_2, c_3, ... vanish, and this expansion becomes illusory.

Example 2. Let $f(z)$ be a function of z, which is regular for all real values of z between $z=0$ and $z=\pi$, and which is zero at $z=0$ and $z=\pi$. Prove that if $f(z)$ is expanded in a sine series, valid between $z=0$ and $z=\pi$, the series will be absolutely and uniformly convergent for all real values of z.

Example 3. $f(z)$ is a function of z which is regular for all real values of z between 0 and π. Prove that if it is expanded in a cosine series, valid between $z=0$ and $z=\pi$, the series will be absolutely and uniformly convergent for all real values of z.

85. *Determination of points of discontinuity.*

The expressions for a_m and b_m which have been found in the last paragraph can be applied to determine the points at which the sum of a given Fourier series is discontinuous. This can best be shewn by an example.

Example. Let it be required to determine the places at which the sum of the series

$$\sin z+\tfrac{1}{3}\sin 3z+\tfrac{1}{5}\sin 5z+\dots$$

is discontinuous.

For this series we have

$$a_m=0,$$

$$b_m=\frac{1-\cos m\pi}{2m}.$$

Comparing this with the formula found in the last paragraph, we have

$$A=0,\quad B=\tfrac{1}{2}-\tfrac{1}{2}\cos m\pi,$$

$$A'=B'=a_m''=b_m''=0.$$

Hence if k_1, k_2 ... are the places at which the analytic character of the sum is broken, we have

$$0=A=\frac{1}{\pi}[\sin mk_1\{f(k_1-0)-f(k_1+0)\}+\sin mk_2\{f(k_2-0)-f(k_2+0)\}+\dots].$$

Since this is true for all values of m, the quantities k_1, k_2, ... must be multiples of π; but

there is only one multiple of π in the range $0<z<2\pi$, namely π itself. So $k_1=\pi$, and $k_2, k_3, \ldots$ do not exist. Substituting $k_1=\pi$ in the equation $B=\frac{1}{2}-\frac{1}{2}\cos m\pi$, we have

$$\tfrac{1}{2}-\tfrac{1}{2}\cos m\pi=-\frac{1}{\pi}[-f(+0)+\cos m\pi\,\{f(\pi-0)-f(\pi+0)\}+f(2\pi-0)].$$

Since this is true for all values of m, we have

$$\tfrac{1}{2}=-\frac{1}{\pi}\{f(2\pi-0)-f(+0)\},$$

and
$$-\tfrac{1}{2}=-\frac{1}{\pi}\{f(\pi-0)-f(\pi+0)\}.$$

This shews that there is a discontinuity at the point $z=\pi$, such that

$$f(\pi-0)-f(\pi+0)=\frac{\pi}{2},$$

and that

$$f(2\pi-0)-f(+0)=-\frac{\pi}{2}.$$

Example. Find the discontinuities in value of the sum of the series

$$\sin z-\tfrac{1}{2}\sin 2z+\tfrac{1}{4}\sin 4z-\tfrac{1}{5}\sin 5z+\tfrac{1}{7}\sin 7z-\tfrac{1}{8}\sin 8z+\tfrac{1}{10}\sin 10z+\ldots.$$

86. *The uniqueness of the Fourier expansion.*

We have seen that if $f(z)$ is a quantity depending on z, and satisfying certain conditions as to finiteness, etc., then the series

$$a_0+\sum_{m=1}^{\infty}(a_m\cos mz+b_m\sin mz),$$

where
$$a_m=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos mt\,dt \qquad (m>1),$$

$$b_m=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin mt\,dt,$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(t)\,dt,$$

has the sum $f(z)$ when $0\leqslant z\leqslant 2\pi$, except at the isolated points at which $f(z)$ is discontinuous.

The question arises whether any other expansion

$$c_0+\sum_{m=1}^{\infty}(c_m\cos mz+d_m\sin mz)$$

of the same form exists, which also represents $f(z)$ in the interval from 0 to 2π; in other words, whether the Fourier expansion is *unique.*

We may observe that it is certainly possible to have other trigonometrical expansions of (say) the form

$$a_0+\sum_{m=1}^{\infty}\left(a_m\cos\frac{mz}{2}+\beta_m\cos\frac{mz}{2}\right)$$

which represent $f(z)$ between 0 and 2π; for write $z=2\zeta$, and consider a function $\phi(\zeta)$, which is such that $\phi(\zeta)=f(2\zeta)$ when $0<\zeta<\pi$, and $\phi(\zeta)=g(\zeta)$ when $\pi<\zeta<2\pi$, where $g(\zeta)$ is any other function. Then on expanding $\phi(\zeta)$ in a Fourier expansion of the form

$$a_0+\sum_{m=0}^{\infty}(\alpha_m \cos m\zeta+\beta_m \cos m\zeta),$$

this expansion represents $f(z)$ when $0<z<2\pi$; and clearly by choosing the function $g(\zeta)$ in different ways an infinite number of such expansions can be obtained.

The question now at issue is, whether other series proceeding in sines and cosines of *integral* multiples of z exist, which differ from Fourier's expansion and yet represent $f(z)$ between 0 and 2π.

If it were possible to have a distinct expansion

$$f(z)=c_0+\sum_{m=1}^{\infty}(c_m \cos mz+d_m \sin mz),$$

then on subtracting this from the Fourier expansion we should have an expansion

$$(a_0-c_0)+\sum_{m=1}^{\infty}\{(a_m-c_m)\cos mz+(b_m-d_m)\sin mz\}$$

whose sum is zero for all values of z between 0 and 2π, except possibly a certain finite number of values (namely the discontinuities).

The investigation therefore turns on the question whether it is possible for such an expansion as this last to exist. We shall shew that it cannot exist, and that consequently the Fourier expansion is unique*.

Let
$$A_0=\tfrac{1}{2}a_0,$$
$$A_m=a_m\cos mz+b_m\sin mz \qquad (m>1);$$
and let
$$\Sigma=A_0+A_1+\ldots+A_m+\ldots$$

be a convergent (not necessarily absolutely convergent) series for values of z from 0 to 2π, so that the limit of a_n and b_n is zero for $n=\infty$; and suppose that (except at certain exceptional points) its sum is zero.

Then the series
$$F(z)=A_0\frac{z^2}{2}-A_1-\frac{A_2}{4}-\ldots-\frac{A_m}{m^2}-\ldots$$

converges absolutely and uniformly for this range of values of z, as is seen by comparing it with the series $\Sigma\frac{1}{m^2}$.

We shall first establish a lemma due to Riemann†, which may be stated thus:

* The proof is due to G. Cantor, *Journal für Math.* LXXII.

† *Collected Works*, p. 213.

The quantity

$$R = \frac{F(z+2\alpha) + F(z-2\alpha) - 2F(z)}{4\alpha^2}$$

tends to the limit $f(z)$ as α tends to zero, if at z the series Σ converges to the sum $f(z)$.

For the term involving a_n in R is

$$-\frac{1}{4\alpha^2 n^2}\{a_n \cos n(z+2\alpha) + a_n \cos n(z-2\alpha) - 2a_n \cos nz\},$$

or

$$\frac{a_n \cos nz \sin^2 n\alpha}{n^2\alpha^2},$$

and similarly the term involving b_n is $\dfrac{b_n \sin nz \sin^2 n\alpha}{n^2\alpha^2}$.

As $F(z)$ converges absolutely, we can rearrange the order of the terms, and so can write

$$R = A_0 + A_1\left(\frac{\sin\alpha}{\alpha}\right)^2 + A_2\left(\frac{\sin 2\alpha}{2\alpha}\right)^2 + \dots.$$

Now considering the series Σ, we can write

$$A_0 + A_1 + A_2 + \dots + A_{n-1} = f(z) + \epsilon_n,$$

say, where z being given, and any small quantity δ being assigned at will, we shall have $|\epsilon_n| < \delta$ for values of $n \geqslant$ some integer m.

Now $A_n = \epsilon_{n+1} - \epsilon_n$ for all values of n.

Therefore substituting, we have

$$R = f(z) + \sum_{n=1}^{\infty} \epsilon_n \left[\left\{\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right\}^2 - \left\{\frac{\sin n\alpha}{n\alpha}\right\}^2\right].$$

Divide the series on the right-hand side of this equation into three parts, for which respectively

(1) $1 \leqslant n \leqslant m$,

(2) $m+1 \leqslant n \leqslant s$, where s is the greatest integer in $\dfrac{\pi}{\alpha}$,

(3) $s+1 \leqslant n$.

The first part consists of a finite number of terms, each tending to zero as α tends to zero, so the first part is zero.

Considering next the second part, the quantities $\dfrac{\sin n\alpha}{n\alpha}$ are of the form $\dfrac{\sin x}{x}$ where $0 < x < \pi$; this quantity decreases as x increases from 0 to π, so the sum of the moduli of the terms in the second part is less than

$$\delta\left[\left(\frac{\sin m\alpha}{m\alpha}\right)^2 - \left(\frac{\sin s\alpha}{s\alpha}\right)^2\right],$$

which tends to zero when δ tends to zero.

Considering next the third part, we can write the nth term in the form

$$\epsilon_n\left[\left(\frac{\sin \overline{n-1}\,\alpha}{\overline{n-1}\,\alpha}\right)^2-\left(\frac{\sin \overline{n-1}\,\alpha}{n\alpha}\right)^2\right]+\frac{\epsilon_n}{n^2\alpha^2}(\sin^2 \overline{n-1}\,\alpha-\sin^2 n\alpha),$$

or

$$\epsilon_n\frac{\sin^2 \overline{n-1}\,\alpha}{\alpha^2}\left[\frac{1}{\overline{n-1}^2}-\frac{1}{n^2}\right]-\epsilon_n\frac{\sin \overline{2n-1}\,\alpha \sin\alpha}{n^2\alpha^2},$$

so, as $|\epsilon_n|<\delta$, its modulus is less than

$$\frac{\delta}{\alpha^2}\left(\frac{1}{\overline{n-1}^2}-\frac{1}{n^2}\right)+\frac{\delta}{n^2\alpha}.$$

Thus the whole sum of the terms in the third part

$$<\frac{\delta}{\alpha^2}\frac{1}{s^2}+\frac{\delta}{\alpha}\left[\frac{1}{\overline{s+1}^2}+\frac{1}{\overline{s+2}^2}+\dots\right]$$

$$<\frac{\delta}{s^2\alpha^2}+\frac{\delta}{\alpha}\int_s^\infty\frac{dx}{x^2}<\frac{\delta}{s^2\alpha^2}+\frac{\delta}{s\alpha}<\frac{\delta}{(\pi-\alpha)^2}+\frac{\delta}{\pi-\alpha}$$

which is ultimately zero. Therefore the three parts of the infinite series in R are all zero; and thus $R=f(z)$ in the limit; which establishes Riemann's lemma.

Next, we shall establish another lemma, due to Schwartz *, which may be stated as follows: *If a and b are two of the exceptional points, so that between $z=a$ and $z=b$ the series Σ converges to the sum zero, then $F(z)$ is a linear function of z between these values.*

For assume that a is less than b, and introduce a function $\phi(z)$, defined by

$$\phi(z)=\theta\left[F(z)-F(a)-\frac{z-a}{b-a}\{F(b)-F(a)\}\right]-\frac{h^2}{2}(z-a)(b-z),$$

where $\theta^2=1$ and h is any constant.

Then substituting in the result of Riemann's lemma, we have

$$\operatorname*{Limit}_{\alpha=0}\frac{\phi(z+\alpha)+\phi(z-\alpha)-2\phi(z)}{\alpha^2}=h^2.$$

Therefore $\phi(z+\alpha)+\phi(z-\alpha)-2\phi(z)$ is positive when α is very small, whatever be the value of z.

Now $\phi(a)=0$ and $\phi(b)=0$. Also $\phi(z)$ is continuous, since $F(z)$ is uniformly convergent, and consequently continuous. Therefore if $\phi(z)$ can be positive between the values a and b of z, it will have a maximum; let this occur at the value c of z.

* Quoted by G. Cantor, *Journal für Math.* LXXII.

Then when α is small, we have

$$\phi(c+\alpha)-\phi(c)<0, \quad \text{and} \quad \phi(c-\alpha)-\phi(c)<0.$$

Adding these relations, we see that the condition just found is violated, and so $\phi(z)$ can *not* be positive at all within the range.

Again, take h small. Choose $\theta = \pm 1$, so choosing the sign that the first term $\theta[F(z)-\ldots]$ is positive. Then $\phi(z)$ is clearly positive, if this first term is not zero.

But $\phi(z)$ is not positive; and thus we must have

$$F(z)-F(a)-\frac{z-a}{b-a}\{F(b)-F(a)\}=0.$$

Therefore $F(z)$ is a linear function of z, which establishes Schwartz's lemma.

We see then that the curve $y=F(z)$ represents a series of straight lines, the beginning and end of each line corresponding to an exceptional point; and as $F(z)$, being uniformly convergent, is a continuous function of z, these lines must form parts of a polygon.

But by Riemann's lemma

$$\operatorname*{Limit}_{\alpha=0} \frac{F(z+\alpha)-F(z)}{\alpha}-\frac{F(z-\alpha)-F(z)}{-\alpha}=0.$$

Now the first of these fractions gives the inclination of the earlier side of the polygon at a vertex and the second of the later; therefore the two sides are continuous in direction, so the equation $y=F(z)$ represents a single line. If then we write $F(z)=cz+c'$, it follows that c and c' have the same values throughout the range. Thus

$$A_0\frac{z^2}{2}-A_1-\ldots-\frac{A_n}{n^2}-\ldots=cz+c';$$

and therefore

$$A_0\frac{z^2}{2}-cz-c'=A_1+\ldots+\frac{A_n}{n^2}+\ldots,$$

the right-hand side of this equation being periodic, with period 2π.

The left-hand side of this equation must therefore be periodic, with period 2π. Thus we have

$$A_0=0, \quad c=0,$$

and
$$-c'=A_1+\ldots+\frac{a_n}{n^2}\cos nz+\frac{b_n}{n^2}\sin nz+\ldots.$$

Now the right-hand side of this equation converges uniformly, so we can

multiply the equation by $\cos nz$, $\sin nz$, respectively, and integrate. This gives

$$\pi \frac{a_n}{n^2} = - c' \int_0^{2\pi} \cos nz dz = 0,$$

and

$$\pi \frac{b_n}{n^2} = - c' \int_0^{2\pi} \sin nz dz = 0.$$

Therefore the a's and b's vanish, so all the coefficients in Σ vanish; which establishes the result that the Fourier expansion is unique.

Miscellaneous Examples.

1. Obtain the expansions

(a) $\dfrac{1 - r\cos z}{1 - 2r\cos z + r^2} = 1 + r\cos z + r^2 \cos 2z + \dots,$

(b) $\dfrac{1}{2}\log(1 - 2r\cos z + r^2) = - r\cos z - \dfrac{1}{2} r^2 \cos 2z - \dfrac{1}{3} r^3 \cos 3z - \dots,$

(c) $\tan^{-1} \dfrac{r \sin z}{1 - r\cos z} = r\sin z + \dfrac{1}{2} r^2 \sin 2z + \dfrac{1}{3} r^3 \sin 3z + \dots,$

(d) $\tan^{-1} \dfrac{2r\sin z}{1 - r^2} = r \sin z + \dfrac{1}{3} r^3 \sin 3z + \dfrac{1}{5} r^5 \sin 5z + \dots,$

and shew that, when $|r| < 1$, they are convergent for all values of z in certain belts parallel to the real axis in the z-plane.

2. Shew that the series

$$\frac{n-1}{2} - \frac{n}{\pi}\left[\frac{\sin \frac{2\pi z}{n}}{1} + \dots + \frac{\sin \frac{(n-1)\,2\pi z}{n}}{n-1} + \frac{\sin (n+1)\frac{2\pi z}{n}}{n+1} + \dots + \frac{\sin \frac{k \,.\, 2\pi z}{n}}{k} + \dots\right],$$

where all the terms for which k is a multiple of n are omitted, represents the greatest integer contained in z, for all real values of z between 0 and n.

3. Shew that the expansions

$$\frac{1}{2}\log\left(2\cos\frac{z}{2}\right) = \cos z - \frac{1}{2}\cos 2z + \frac{1}{3}\cos 3z \dots$$

and

$$\frac{1}{2}\log\left(2\sin\frac{z}{2}\right) = -\cos z - \frac{1}{2}\cos 2z - \frac{1}{3}\cos 3z \dots$$

are valid for all real values of z, except multiples of π.

4. Obtain the expansion

$$\sum_{m=0}^{\infty} \frac{(-1)^m \cos mz}{(m+1)(m+2)} = (\cos z + \cos 2z) \log\left(2\cos\frac{z}{2}\right) + \frac{z}{2}(\sin 2z + \sin z) - \cos z,$$

and find the range of values of z for which it is applicable.

(Trinity College, 1898.)

5. Let n be an integer $\geqslant 2$, and let $x_1, x_2, \ldots x_{n-1}$ be quantities satisfying the conditions

$$0 < x_1 < x_2 < \ldots < x_{n-1} < 1,$$

and write $x_0 = 0$, $x_n = 1$.

Let $c_0, c_1, c_2, \ldots c_{n-1}$ be real arbitrary constants and let a function $\phi(x)$ be defined by the equalities

$$\phi(x) = c_0 + c_1 + \ldots + c_s, \qquad \text{for } x_s < x < x_{s+1} \qquad (s = 0, 1, 2, \ldots n-1),$$

$$\phi(x) = c_0 \text{ for } x = x_0,$$

$$\phi(x) = c_0 + c_1 + \ldots + c_{s-1} + \frac{c_s}{2}, \qquad \text{for } x = x_s \qquad (s = 1, 2, \ldots n-1),$$

$$\phi(x) = c_0 + c_1 + \ldots + c_{n-1}, \qquad \text{for } x = x_n.$$

Shew that

$$\phi(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos 2m\pi x + b_m \sin 2m\pi x), \qquad \text{for } 0 < x < 1,$$

and

$$\frac{\phi(0) + \phi(1)}{2} = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m,$$

where the coefficients a_m and b_m are given by

$$a_0 = 2 \sum_{r=0}^{n-1} c_r (1 - x_r),$$

$$a_m = -\frac{1}{m\pi} \sum_{r=0}^{n-1} c_r \sin 2m\pi x_r, \qquad \text{for } m \geqslant 1,$$

$$b_m = -\frac{1}{m\pi} \sum_{r=0}^{n-1} c_r (1 - \cos 2m\pi x_r) \qquad \text{for } m \geqslant 1.$$

(Berger.)

6. Shew that between the values $-\pi$ and $+\pi$ of z the following expansions hold:

$$\sin mz = \frac{2}{\pi} \sin m\pi \left(\frac{\sin z}{1^2 - m^2} - \frac{2 \sin 2z}{2^2 - m^2} + \frac{3 \sin 3z}{3^2 - m^2} - \ldots \right),$$

$$\cos mz = \frac{2}{\pi} \sin m\pi \left(\frac{1}{2m} + \frac{m \cos z}{1^2 - m^2} - \frac{m \cos 2z}{2^2 - m^2} + \frac{m \cos 3z}{3^2 - m^2} - \ldots \right),$$

$$\frac{e^{mz} + e^{-mz}}{e^{m\pi} - e^{-m\pi}} = \frac{2}{\pi} \left(\frac{1}{2m} - \frac{m \cos z}{1^2 + m^2} + \frac{m \cos 2z}{2^2 + m^2} - \frac{m \cos 3z}{3^2 + m^2} + \ldots \right).$$

7. Obtain the expansions

$$\sum_{m=-\infty}^{\infty} \frac{\sin \mu (z + m\pi)}{z + m\pi} = \begin{cases} \dfrac{\sin (2n+1) z}{\sin z} & (2n < \mu < 2n+2) \\[2ex] \sin 2nz \cot z & (\mu = 2n) \end{cases},$$

and

$$\sum_{m=-\infty}^{\infty} \frac{\cos \mu (z + m\pi)}{z + m\pi} = \begin{cases} \dfrac{\cos (2n+1) z}{\sin z} & (2n < \mu < n+2) \\[2ex] \cos 2nz \cot z & (\mu = 2n) \end{cases}.$$

If p and q are positive integers, shew that

$$\sum_{m=-\infty}^{\infty} \frac{\sin (qm+p)\dfrac{2n\pi}{q}}{qm+p} = \frac{\pi}{q} \sin \frac{2np\pi}{q} \cot \frac{p\pi}{q},$$

$$\sum_{m=-\infty}^{\infty} \frac{\cos (qm+p)\dfrac{2n\pi}{q}}{qm+p} = \frac{\pi}{q} \cos \frac{2np\pi}{q} \cot \frac{p\pi}{q}.$$

8. Prove that the locus represented by

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny = 0$$

is two systems of lines at right angles, dividing the coordinate plane into squares of area π^2.

(Cambridge Mathematical Tripos, Part I., 1895.)

9. If m is an integer, shew that

$$\cos^{2m} z = 2 . \frac{1.3.5 \ldots (2m-1)}{2.4.6 \ldots 2m} \left\{ \frac{1}{2} + \frac{m}{m+1} \cos 2z + \frac{m(m-1)}{(m+1)(m+2)} \cos 4z + \frac{m(m-1)(m-2)}{(m+1)(m+2)(m+3)} \cos 6z + \ldots \right\}$$

(a terminating series),

$$\cos^{2m-1} z = \frac{4}{\pi} \frac{2.4.6 \ldots (2m-2)}{1.3.5 \ldots (2m-1)} \left\{ \frac{1}{2} + \frac{2m-1}{2m+1} \cos 2z + \frac{(2m-1)(2m-3)}{(2m+1)(2m+3)} \cos 4z + \ldots \right\}$$

(an infinite series).

Shew also that

$$1 = \frac{4}{\pi} \left(\cos z - \frac{\cos 3z}{3} + \frac{\cos 5z}{5} - \frac{\cos 7z}{7} + \frac{\cos 9z}{9} - \ldots \right),$$

and

$$\cos^3 z = \frac{8}{\pi} \left(\frac{\cos z}{1.3} + \frac{\cos 3z}{1.3.5} - \frac{\cos 5z}{3.5.7} + \frac{\cos 7z}{5.7.9} - \frac{\cos 9z}{7.9.11} + \ldots \right).$$

10. A point moves in a straight line with a velocity which is initially u, and which receives constant increments, each equal to u, at equal intervals τ. Prove that the velocity at any time t after the beginning of the motion is

$$\frac{u}{2} + \frac{ut}{\tau} + \frac{u}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi t}{\tau},$$

and that the distance traversed is

$$\frac{ut}{2\tau}(t+\tau) + \frac{u\tau}{12} - \frac{u\tau}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \frac{2m\pi t}{\tau}.$$

11. Shew that

$$\sin (a + 2ux\pi) = \frac{\sin x\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n \sin (a + 2nv\pi)}{x-n},$$

where u is the difference between the real quantity v (supposed not to be an odd multiple of $\frac{1}{2}$) and the integer to which v is most nearly equal.

(Cambridge Mathematical Tripos, Part II., 1895.)

12. Let n be an integer $\geqslant 3$, and let $g_0, g_1, g_2, \ldots$ be an infinite set of quantities, which satisfy the conditions,

$$g_0 = 0,$$

$$g_1 + g_2 + g_3 + \ldots + g_{n-1} = 0,$$

$$g_{r+n} = g_r, \text{ for } r \geqslant 0.$$

Let x be a real variable, and let s be the greatest integer contained in nx.

Shew that when $x \geqslant 0$,

$$\sum_{r=0}^{s} g_r = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos 2m\pi x + b_m \sin 2m\pi x),$$

if r is not a multiple of $\frac{1}{n}$;

but

$$\sum_{r=0}^{s} g_r - \frac{g_s}{2} = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos 2m\pi x + b_m \sin 2m\pi x),$$

if r is a multiple of $\frac{1}{n}$; the coefficients a_m and b_m being determined by the formulae

$$a_0 = -\frac{2}{n} \sum_{r=1}^{n-1} g_r,$$

$$a_m = -\frac{1}{m\pi} \sum_{r=1}^{n-1} g_r \sin \frac{2mr\pi}{n} \qquad (m \geqslant 1),$$

$$b_m = \frac{1}{m\pi} \sum_{r=1}^{n-1} g_r \cos \frac{2mr\pi}{n} \qquad (m \geqslant 1). \qquad \text{(Berger.)}$$

13. Let x be a real variable between 0 and 1, and let n be an integer $\geqslant 5$, of the form $4m+1$, where m is an integer.

Let $E(a)$ denote the greatest integer contained in a.

Shew that

$$(-1)^{E\left(\frac{nx}{2}\right)} + (-1)^{E\left(\frac{nx+1}{2}\right)} = \frac{2}{n} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \tan \frac{2m\pi}{n} \cos 2m\pi x,$$

if x is not a multiple of $\frac{1}{n}$;

but

$$\sin \frac{n\pi x}{2} + \frac{\cos n\pi x}{2} = \frac{2}{n} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \tan \frac{2m\pi}{n} \cos 2m\pi x,$$

if x is a multiple of $\frac{1}{n}$. (Berger.)

14. Let x be a real variable between 0 and 1, and let n be an odd number $\geqslant 3$. Shew that

$$(-1)^s = \frac{1}{n} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \tan \frac{m\pi}{n} \cos 2m\pi x,$$

if x is not a multiple of $\frac{1}{n}$, where s is the greatest integer contained in nx; but

$$0=\frac{1}{n}+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\tan\frac{m\pi}{n}\cos 2m\pi x,$$

if x is a multiple of $\frac{1}{n}$. (Berger.)

15. Let x denote a real variable between 0 and 1, and let n be an integer $\geqslant 3$; further, let $E(a)$ be the greatest integer contained in a. Shew that

$$\frac{E(nx)\{E(nx)+1-n\}}{n}=-\frac{(n-1)(n-2)}{6n}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\cot\frac{m\pi}{n}\cos 2m\pi x,$$

if x is not a multiple of $\frac{1}{n}$; but

$$nx^2-nx+\frac{1}{2}=-\frac{(n-1)(n-2)}{6n}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\cot\frac{m\pi}{n}\cos 2m\pi x,$$

if x is a multiple of $\frac{1}{n}$. (Berger.)

16. Assuming the possibility of expanding $f(x)$ in a series of the form $\Sigma A_k \sin kx$, where k is a root of the equation $k\cos ak+b\sin ak=0$, and the summation is extended to all positive roots of this equation, determine the constants A_k.

(Cambridge Mathematical Tripos, Part I., 1898.)

17. If

$$\frac{ae^{ax}}{e^a-1}=\sum_{n=0}^{\infty}\frac{a^n V_n(x)}{n!},$$

shew that

$$\cos 2\pi x+\frac{\cos 4\pi x}{2^{2n}}+\frac{\cos 6\pi x}{3^{2n}}+\ldots=(-1)^{n-1}\frac{2^{2n-1}\pi^{2n}}{2n!}V_{2n}(x),$$

$$\sin 2\pi x+\frac{\sin 4\pi x}{2^{2n+1}}+\frac{\sin 6\pi x}{3^{2n+1}}+\ldots=(-1)^{n+1}\frac{2^{2n}\pi^{2n+1}}{2n+1!}V_{2n+1}(x).$$

(Cambridge Mathematical Tripos, Part II., 1896.)

18. If

$$f(x)=\tfrac{1}{2}a_0+a_1\cos x+a_2\cos 2x+\ldots,$$

shew that

$$a_n=\frac{4}{\pi}\int_0^{\infty}f(x)\cos nx\tan\tfrac{1}{2}x\,\frac{dx}{x}.$$

If

$$\phi(x)=b_1\sin x+b_2\sin 2x+\ldots,$$

shew that

$$b_n=\frac{4}{\pi}\int_0^{\infty}\phi(x)\sin nx\tan\tfrac{1}{2}x\,\frac{dx}{x}.$$ (Beau.)

19. Prove that the series $\quad \sum_1^{\infty}A_n\sin\frac{n\pi x}{a},$

where $\quad A_n=\frac{2}{a}\int_0^a\sin\frac{n\pi v}{a}f(v)\,dv,$

is equal to $f(x)$ for any value of x lying between 0 and a about which $f(x)$ is continuous.

If $f(0)$, $f(a)$ are the limits of $f(\epsilon)$, $f(a-\epsilon)$, when the positive quantity ϵ diminishes to zero, and if $f(x)$ has sudden increases of value h, k, corresponding to the values a, β, ... of x, the limit for $n=\infty$ of nA_n can be written in the form

$$\frac{2}{\pi}\left\{f(0)-(-1)^n f(a)+h\cos\frac{n\pi a}{a}+k\cos\frac{n\pi\beta}{a}+\ldots\right\}.$$

Shew that the series

$$\sin 3x+\frac{1}{3}\sin 9x+\frac{1}{5}\sin 15x+\ldots-2\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\right)$$
$$+\frac{3\sqrt{3}}{\pi}\left\{\sin x-\frac{1}{5^2}\sin 5x+\frac{1}{7^2}\sin 7x-\frac{1}{11^2}\sin 11x+\ldots\right\}$$

has the limit $-\frac{1}{4}\pi$ when x, lying between 0 and π, approaches indefinitely near to one or other value, and that it has sudden changes of value $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ corresponding to the values $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$ of x.

(Cambridge Mathematical Tripos, Part I., 1893.)

20. If, for all real values of x,

$$F(x)=A_0+A_1\cos x+A_2\cos 2x+A_3\cos 3x+\ldots,$$

then

$$\text{(i)}\quad \int_0^\infty \cos\left(\frac{x^2}{2w}\right)F(x)\,dx=(U+V)\left(\frac{w\pi}{2}\right)^{\frac{1}{2}},$$

$$\text{(ii)}\quad \int_0^{(4n+1)}\cos\left(\frac{x^2}{4w}\right)F(x)\,dx=\int_0^\pi XF(x)\,dx,$$

where

$$U=A_0+A_1\cos w+A_2\cos 4w+A_3\cos 9w+\ldots,$$

$$V=A_1\sin w+A_2\sin 4w+A_3\sin 9w+\ldots,$$

$$X=\cos\frac{x^2}{4w}+2\cos\frac{4\pi^2+x^2}{4w}\cos\frac{\pi x}{w}+2\cos\frac{4(2\pi)^2+x^2}{4w}\cos\frac{2\pi x}{w}+\ldots$$
$$+2\cos\frac{4(2n\pi)^2+x^2}{4w}\cos\frac{2n\pi x}{w}.$$

Prove these formulae, and thence deduce the result

$$(U+V)\left(\frac{\pi}{2w}\right)^{\frac{1}{2}}=\frac{1}{2}F(0)+F(w)\cos\frac{1^2w}{4}+F(2w)\cos\frac{2^2w}{4}+\ldots$$
$$+\frac{1}{2}F(0)\cos\frac{\pi^2}{w}-F(w)\cos\left(\frac{\pi^2}{w}+\frac{1^2w}{4}\right)+F(2w)\cos\left(\frac{\pi^2}{w}+\frac{2^2w}{4}\right)-\ldots,$$

where $w=\frac{2\pi}{k}$, k being a positive integer. When k is even, the last term of each series involves $F(\frac{1}{2}kw)$ and is to be multiplied by $\frac{1}{2}$; when k is uneven, the last term involves $F\{\frac{1}{2}(k-1)w\}$.

(Cambridge Mathematical Tripos, Part II., 1896.)

CHAPTER VIII.

ASYMPTOTIC EXPANSIONS.

87. *Simple example of an asymptotic expansion.*

Consider the function

$$f(x) = \int_x^\infty \frac{e^{x-t}\,dt}{t},$$

where x is real and positive, and the path of integration is the real axis in the t-plane.

Integrating by parts, we have

$$f(x) = \frac{1}{x} - \int_x^\infty \frac{e^{x-t}\,dt}{t^2},$$

and by repeated integration by parts, we obtain

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \dots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n\, n! \int_x^\infty \frac{e^{x-t}\,dt}{t^{n+1}}.$$

In connexion with the function $f(x)$, we therefore consider the series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \dots + \frac{(-1)^{n-1}(n-1)!}{x^n} + \dots.$$

We shall denote this series by S, and shall write

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \dots + \frac{(-1)^n\, n!}{x^{n+1}} = S_n.$$

The ratio of the nth term of the series S to the $(n-1)$th term is $-\dfrac{n-1}{x}$; for values of n greater than $1+x$, this is greater than unity. *The series S is therefore divergent for all values of x.* In spite of this, however, the series can under certain circumstances be used for the calculation of $f(x)$; this can be seen in the following way.

Take any definite value for the number n, and calculate the value of S_n. We have

$$f(x) - S_n = (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{x-t}\,dt}{t^{n+2}},$$

and therefore

$$|f(x) - S_n| = (n+1)! \int_x^\infty \frac{e^{x-t}\,dt}{t^{n+2}}$$

$$< (n+1)! \int_x^\infty \frac{dt}{t^{n+2}}, \text{ since } e^{x-t} < 1 \text{ and } t \text{ is positive,}$$

$$< \frac{n!}{x^{n+1}}.$$

For values of x which are sufficiently large, the right-hand member of this equation is very small. Thus if we take $x > 2n$, we have

$$|f(x) - S_n| < \frac{1}{2^{n+1}n^2},$$

which for large values of n is very small. It follows therefore that *the value of the function $f(x)$ can be calculated with great accuracy for large values of x, by taking the sum of a finite number of terms of the series S.*

The series is on this account said to be an *asymptotic expansion* of the function $f(x)$. The precise definition of an asymptotic expansion will now be given.

88. *Definition of an asymptotic expansion.*

A divergent series

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots + \frac{A_n}{x^n} + \ldots,$$

in which the sum of the $(n+1)$ first terms is S_n, is said to be an *asymptotic expansion* of a function $f(x)$, if the expression $x^n\{f(x) - S_n\}$ tends to zero as x (supposed for the present to be real and positive) increases indefinitely. When this is the case, if x is sufficiently great, we have

$$x^n(f - S_n) = \epsilon,$$

where ϵ is very small; and the error $\frac{\epsilon}{x^n}$ committed in taking for $f(x)$ the $(n+1)$ first terms of the series is very small. This error is in fact infinitesimal compared with the error committed in taking for $f(x)$ the n first terms of the series: for this latter error is

$$\frac{A_n + \epsilon}{x^n},$$

and ϵ is in general infinitely small compared with $A_n + \epsilon$.

The definition which has just been given is due to Poincaré*. Special asymptotic expansions had, however, been discovered and used in the eighteenth century by Stirling, Maclaurin and Euler. Asymptotic expansions are of great importance in the theory of Linear Differential Equations, and in Dynamical Astronomy; these applications are, however, outside the scope of the present work, and for them reference may be made to Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, and the second volume of Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*.

The example discussed in the preceding article clearly satisfies the definition just given: for

$$|x^n\{f(x)-S_n\}|<\frac{n!}{x},$$

and the right-hand member of this equation tends to zero as x tends to infinity.

The term "asymptotic expansion" is sometimes used in a somewhat wider sense; if F, ϕ, and J are three functions of x, and if a series

$$A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots$$

is the asymptotic expansion of the function

$$\frac{J-\phi}{F},$$

we can say that the series

$$\phi+FA_0+\frac{FA_1}{x}+\frac{FA_2}{x^2}+\dots$$

is an asymptotic expansion of the function J.

For the sake of simplicity, we shall consider asymptotic expansions only in connexion with real positive values of the argument. The theory for complex values of the argument may be discussed by an extension of the analysis.

89. *Another example of an asymptotic expansion.*

As a second example, consider the function $f(x)$, represented by the series

$$f(x)=\sum_{k=1}^{\infty}\frac{c^k}{x+k}\quad\dots\dots\dots(1),$$

where c is a positive constant less than unity.

The ratio of the kth term of this series to the $(k-1)$th is less than unity when k is large, except when x is a negative integer, and conse-

* *Acta Mathematica*, VIII. (1886), pp. 295—344.

quently the series converges for all values of x except negative integral values. We shall confine our attention to positive values of x. We have, when $x > k$,

$$\frac{1}{x+k} = \frac{1}{x} - \frac{k}{x^2} + \frac{k^2}{x^3} - \frac{k^3}{x^4} + \frac{k^4}{x^5} - \dots.$$

If, therefore, it were allowable to expand each fraction $\frac{1}{x+k}$ in this way, and to rearrange the series (1) according to descending powers of x, we should obtain the series

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \dots + \frac{A_n}{x^n} + \dots \quad \dots\dots\dots\dots\dots\dots(2),$$

where
$$A_1 = \sum_{k=1}^{\infty} c^k; \quad A_2 = - \sum_{k=1}^{\infty} kc^k, \text{ etc.}$$

But this procedure is not legitimate, and in fact the series (2) diverges. We can, however, shew that the series (2) is an asymptotic expansion of $f(x)$, which will enable us to calculate $f(x)$ for large values of x.

For let
$$S_n = \frac{A_1}{x} + \frac{A_2}{x^2} + \dots + \frac{A_{n+1}}{x^{n+1}}.$$

Then
$$S_n = \sum_{k=1}^{\infty} \left(\frac{c^k}{x} - \frac{kc^k}{x^2} + \frac{k^2c^k}{x^3} + \dots + \frac{(-1)^n k^n c^k}{x^{n+1}} \right)$$

$$= \sum_{k=1}^{\infty} \left\{ 1 - \left(-\frac{k}{x} \right)^{n+1} \right\} \frac{c^k}{x+k};$$

so
$$f(x) - S_n = \sum_{k=1}^{\infty} \left(-\frac{k}{x} \right)^{n+1} \frac{c^k}{x+k},$$

and
$$x^n \{ f(x) - S_n \} = \frac{(-1)^{n+1}}{x} \sum_{k=1}^{\infty} \frac{k^{n+1} c^k}{x+k}.$$

Now $\sum_{k=1}^{\infty} \frac{k^{n+1}c^k}{x+k}$ is finite, and so when x is infinitely great the right-hand member is infinitesimal.

Therefore $x^n \{f(x) - S_n\}$ tends to zero when x tends to infinity; and so the series (2) is an asymptotic expansion of $f(x)$.

Example. If $f(x) = \int_x^{\infty} e^{x^2 - t^2} dt$, where x is supposed to be real and positive and the path of integration is real, prove that the divergent series

$$\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1.3}{2^3 x^5} - \frac{1.3.5}{2^4 x^7} + \dots$$

is the asymptotic expansion of $f(x)$.

90. *Multiplication of asymptotic expansions.*

We shall now shew that two asymptotic expansions can be multiplied together in the same way as ordinary series, the result being a new asymptotic expansion.

For suppose that

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots + \frac{A_n}{x^n} + \ldots,$$

$$A_0' + \frac{A_1'}{x} + \frac{A_2'}{x^2} + \ldots + \frac{A_n'}{x^n} + \ldots$$

are asymptotic expansions representing functions $J(x)$ and $J'(x)$ respectively, and let S_n and S_n' be the sums of their $(n+1)$ first terms; so that

$$\left.\begin{array}{l} \underset{x=\infty}{\text{Limit}}\ x^n (J - S_n) = 0 \\ \underset{x=\infty}{\text{Limit}}\ x^n (J' - S_n') = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Form the product of the two series in the ordinary way; let it be

$$B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots + \frac{B_n}{x^n} + \ldots,$$

and let Σ_n be the sum of its n first terms.

As S_n, S_n' and Σ_n are simply polynomials in $\frac{1}{x}$, we have clearly

$$\underset{x=\infty}{\text{Limit}}\ x^n (S_n S_n' - \Sigma_n) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now by (1), we can write

$$J = S_n + \frac{\epsilon}{x^n},$$

$$J' = S_n' + \frac{\epsilon'}{x^n},$$

where $\underset{x=\infty}{\text{Limit}}\ \epsilon = 0, \qquad \underset{x=\infty}{\text{Limit}}\ \epsilon' = 0.$

Then $$x^n (JJ' - S_n S_n') = S_n'\epsilon + S_n\epsilon' + \frac{\epsilon\epsilon'}{x^n}.$$

The terms in the right-hand member tend to zero as x tends to infinity. Hence

$$\underset{x=\infty}{\text{Limit}}\ x^n (JJ' - S_n S_n') = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3) we have

$$\operatorname*{Limit}_{x=\infty} x^n (JJ' - \Sigma_n) = 0,$$

and therefore the series

$$B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots$$

is the asymptotic expansion of the function JJ'.

91. *Integration of asymptotic expansions.*

We shall now shew that it is permissible to integrate an asymptotic expansion term by term, the resulting series being the asymptotic expansion of the function represented by the original series.

For let the series

$$\frac{A_2}{x^2} + \frac{A_3}{x^3} + \dots + \frac{A_n}{x^n} + \dots$$

represent the function $J(x)$ asymptotically, and let S_n denote the sum

$$\frac{A_2}{x^2} + \frac{A_3}{x^3} + \dots + \frac{A_n}{x^n}.$$

Then, however small a real positive constant quantity ϵ may be taken, it is possible to choose x so large that

$$|J(x) - S_n| < \frac{\epsilon}{x^n},$$

and therefore

$$\left| \int_x^\infty J(x)\, dx - \int_x^\infty S_n dx \right| \leqslant \int_x^\infty |J(x) - S_n|\, dx$$

$$< \frac{\epsilon}{(n-1)\, x^{n-1}},$$

and therefore the integrated series

$$\frac{A_2}{x} + \frac{A_3}{2x^2} + \dots + \frac{A_n}{(n-1)\, x^{n-1}} + \dots$$

is the asymptotic expansion of the function

$$\int_x^\infty J(x)\, dx.$$

On the other hand, it is not in general permissible to differentiate an asymptotic expansion.

92. *Uniqueness of an asymptotic expansion.*

A question naturally suggests itself, as to whether a given series can be the asymptotic expansion of several distinct functions. The answer to this

is in the affirmative. To shew this, we first observe that there are functions $L(x)$ which are represented asymptotically by a series all of whose terms are zero, i.e. functions such that

$$\operatorname*{Limit}_{x=\infty} x^n L(x) = 0,$$

whatever n may be, when x (supposed to be real and positive) increases indefinitely. The function e^{-x} is in fact such a function. The asymptotic expansion of a function $J(x)$ is therefore also the asymptotic expansion of

$$J(x) + L(x).$$

On the other hand, a function cannot be represented by more than one distinct asymptotic expansion for real positive values of x; for if

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots$$

and

$$B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots$$

are two asymptotic expansions of the same function, then

$$\operatorname*{Limit}_{x=\infty} x^n \left(A_0 + \frac{A_1}{x} + \dots + \frac{A_n}{x^n} - B_0 - \frac{B_1}{x} - \dots - \frac{B_n}{x^n}\right) = 0,$$

which can only be if $\qquad A_0 = B_0;\ A_1 = B_1$, etc.

Important examples of asymptotic expansions will be discussed later, in connexion with the Gamma-function and the Bessel functions.

Miscellaneous Examples.

1. Shew that the series

$$\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \dots$$

is the asymptotic expansion of the function

$$\int_0^x \frac{e^{t-x} - e^{-x}}{t}\, dt$$

when x is real and positive.

2. Discuss the representation of the function

$$f(x) = \int_{-\infty}^{0} \phi(t)\, e^{tx}\, dt$$

(where x is supposed real and positive, and ϕ is an arbitrary function of its argument) by means of the series

$$f(x) = \frac{\phi(0)}{x} - \frac{\phi'(0)}{x^2} + \frac{\phi''(0)}{x^3} - \dots.$$

Shew that in certain cases (e.g. $\phi(t)=e^{at}$) the series is absolutely convergent, and represents $f(x)$ for large positive values of x; but that in certain other cases the series is the asymptotic expansion of $f(x)$.

3. Shew that the divergent series

$$\frac{1}{z}+\frac{a-1}{z^2}+\frac{(a-1)(a-2)}{z^3}+\dots$$

is the asymptotic expansion of the function

$$\frac{z^{-a}}{\log z}\int_z^\infty e^{-x}x^{a-1}dx$$

for large positive values of z.

4. Shew that the function

$$f(x)=\int_0^\infty \left\{\log u+\log\left(\frac{1}{1-e^{-u}}\right)\right\}\frac{du}{u}e^{-xu}$$

has the asymptotic expansion

$$f(x)=\frac{1}{2x}-\frac{B_1}{2^2x^2}+\frac{B_3}{4^2x^4}-\frac{B_5}{6^2x^6}+\dots,$$

where B_1, B_3, ... are Bernoulli's numbers.

Shew also that $f(x)$ can be developed as an absolutely convergent series of the form

$$f(x)=\sum_{k=1}^\infty \frac{c_k}{(x+1)(x+2)\dots(x+k)}.$$ (Schlömilch.)

5. Shew that the function

$$\phi(a)=\int_0^a e^{x^2-a^2}dx$$

has the asymptotic expansion

$$\sum_{n=1}^\infty \frac{1.3\dots(2n-3)}{2^n a^{2n-1}}.$$

PART II.

THE TRANSCENDENTAL FUNCTIONS.

CHAPTER IX.

THE GAMMA-FUNCTION.

93. *Definition of the Gamma-function: Euler's form.*

Consider the infinite product

$$\frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}.$$

This product clearly diverges if z is a negative integer, for then one of the denominator-factors vanishes. If z is not a negative integer, the product will (§ 23) be absolutely convergent, provided the series

$$\sum_{n=1}^{\infty}\left\{z\log\left(1+\frac{1}{n}\right)-\log\left(1+\frac{z}{n}\right)\right\}$$

is absolutely convergent; but since when n is large we have

$$z\log\left(1+\frac{1}{n}\right)=\frac{z}{n}-\frac{z}{2n^2}+\dots$$

and

$$\log\left(1+\frac{z}{n}\right)=\frac{z}{n}-\frac{z^2}{2n^2}+\dots,$$

the terms of this series ultimately bear a finite value to the terms of the series

$$\sum_{n=1}^{\infty}\frac{z^2-z}{2n^2},$$

and therefore to the terms of the series $\Sigma\,\frac{1}{n^2}$, which is absolutely convergent. The infinite product is therefore absolutely convergent for all values of z, except negative integral values.

This product may be regarded as the definition of a new function of the variable z; we shall call it the *Gamma-function*, and denote it by $\Gamma(z)$, so that

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}.$$

This form of the function was first given by Euler; but the notation $\Gamma(z)$ is due to Legendre, who applied it in 1814 to an integral which will presently be discussed, and which represents the Gamma-function in some cases.

Example. Prove that

$$\Gamma(z) = \operatorname*{Limit}_{n=\infty} \frac{1 \cdot 2 \ldots (n-1)}{z(z+1)\ldots(z+n-1)} n^z. \qquad \text{(Gauss.)}$$

94. *The Weierstrassian form for the Gamma-function.*

Another form of the Gamma-function can be obtained as follows:

We have

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{e^{z \log \frac{n+1}{n}}}{1+\frac{z}{n}}$$

$$= \frac{1}{z} \operatorname*{Limit}_{m=\infty} e^{z \log(m+1)} \prod_{n=1}^{m} \frac{1}{\left(1+\frac{z}{n}\right)}$$

$$= \frac{1}{z} \operatorname*{Limit}_{m=\infty} e^{z\left\{\log(m+1) - 1 - \frac{1}{2} - \ldots - \frac{1}{m}\right\}} \prod_{n=1}^{m} \frac{e^{\frac{z}{n}}}{1+\frac{z}{n}}.$$

Now $$1 + \frac{1}{2} + \ldots + \frac{1}{m} - \log(m+1) = \sum_{n=1}^{m} \left(\frac{1}{n} - \log \frac{n+1}{n}\right)$$

$$= \sum_{n=1}^{m} \int_0^1 \left(\frac{1}{n} - \frac{1}{n+x}\right) dx$$

$$= \int_0^1 \left\{\sum_{n=1}^{m} \frac{x}{n(n+x)}\right\} dx.$$

Now the series $\sum_{n=1}^{\infty} \frac{x}{n(n+x)}$ is absolutely and uniformly convergent for real values of x between 0 and 1, as is seen by comparing it with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2};$$

hence as m increases, the right-hand member of this equation tends to the limit

$$\int_0^1 \left\{ \sum_{n=1}^{\infty} \frac{x}{n(n+x)} \right\} dx,$$

which is finite, since the range of integration is finite and the sum of the series $\sum_{n=1}^{\infty} \frac{x}{n(n+x)}$ is finite. This limit is known as Euler's constant, and we shall denote it by γ. Its numerical value is

$$0{\cdot}5772157\ldots.$$

Thus $$\operatorname*{Limit}_{m=\infty} \left\{ 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \log(m+1) \right\} = \gamma,$$

and so $$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \frac{e^{\frac{z}{n}}}{1 + \frac{z}{n}},$$

or $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\}.$$

This form (due to Weierstrass) shews that $\frac{1}{\Gamma(z)}$ is a regular function of z for all values of z.

Example 1. Prove that

$$\Gamma'(1) = -\gamma,$$

where γ is Euler's constant.

For differentiating logarithmically the equation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_1^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\},$$

and putting $z = 1$ after the differentiations have been performed, we have

$$-\Gamma'(1) = 1 + \gamma + \sum_1^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right),$$

or $$\Gamma'(1) = -\gamma.$$

Example 2. Shew that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \int_0^1 \frac{1 - (1-z)^n}{z} dz,$$

and hence that Euler's constant γ is given by

$$\gamma = \int_0^1 \frac{1 - e^{-z} - e^{-\frac{1}{z}}}{z} dz.$$

Example 3. Shew that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{z+n}\right) e^{\frac{x}{n}}$$

has the value

$$\frac{e^{\gamma x}\Gamma(z+1)}{\Gamma(z-x+1)}.$$

For
$$\prod_{n=1}^{\infty}\left(1-\frac{x}{z+n}\right)e^{\frac{x}{n}}=\prod_{n=1}^{\infty}\frac{n+z-x}{n+z}e^{\frac{x}{n}}$$

$$=\prod_{n=1}^{\infty}\frac{(n+z-x)e^{-\frac{z-x}{n}}}{(n+z)e^{-\frac{z}{n}}}$$

$$=\frac{\prod_{n=1}^{\infty}\left(1+\frac{z-x}{n}\right)e^{-\frac{z-x}{n}}}{\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}}.$$

The numerator of this expression is Weierstrass' form of

$$\frac{1}{(z-x)e^{\gamma(z-x)}\Gamma(z-x)},$$

and the denominator is

$$\frac{1}{ze^{\gamma z}\Gamma(z)}.$$

Therefore the given expression has the value

$$\frac{e^{\gamma x}z\Gamma(z)}{(z-x)\Gamma(z-x)}.$$

95. *The difference-equation satisfied by the Gamma-function.*

We shall now shew that the function $\Gamma(z)$ satisfies the difference-equation

$$\Gamma(z+1)=z\Gamma(z).$$

We have

$$\Gamma(z+1)=\frac{1}{z+1}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z+1}}{1+\frac{z+1}{n}}$$

$$=\frac{1}{z+1}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}\frac{n+1}{n}}{\frac{n+z+1}{n}}$$

$$=\frac{1}{z+1}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n+1}}=\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}}$$

$$=z\Gamma(z).$$

This is one of the most characteristic properties of the Gamma-function.

It follows that if z is a positive integer, we have

$$\Gamma(z)=(z-1)!$$

Example. Prove that

$$\frac{1}{\Gamma(z+1)}+\frac{1}{\Gamma(z+2)}+\frac{1}{\Gamma(z+3)}+\dots$$
$$=\frac{e}{\Gamma(z)}\left\{\frac{1}{z}-\frac{1}{1!}\frac{1}{z+1}+\frac{1}{2!}\frac{1}{z+2}-\frac{1}{3!}\frac{1}{z+3}+\dots\right\}.$$

For consider the quantity

$$\frac{1}{z}+\frac{1}{z(z+1)}+\frac{1}{z(z+1)(z+2)}+\dots.$$

This can be expressed as the sum of a number of partial fractions, in the form

$$\frac{a_0}{z}+\frac{a_1}{z+1}+\dots+\frac{a_n}{z+n}+\dots.$$

To find the coefficients a, multiply by $(z+n)$ and put $z=-n$; we thus obtain

$$a_n=\frac{1}{(-1)^n n!}\left\{1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots\right\}=\frac{(-)^n e}{n!}.$$

Therefore

$$\frac{1}{z}+\frac{1}{z(z+1)}+\frac{1}{z(z+1)(z+2)}+\dots=e\left\{\frac{1}{z}-\frac{1}{(z+1)1!}+\frac{1}{(z+2)2!}-\dots\right\}.$$

But

$$\frac{1}{z(z+1)\dots(z+n)}=\frac{\Gamma(z+n+1)}{\Gamma(z)},$$

whence the required result follows.

96. *Evaluation of a general class of infinite products.*

By means of the Gamma-function, it is possible to evaluate the general class of infinite products of the form

$$\prod_{n=1}^{\infty} u_n,$$

where u_n is any rational function of its index n.

For resolving u_n into its factors with respect to n, we can write the infinite product in the form

$$\prod_{n=1}^{\infty}\left\{\frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)\dots(n-b_l)}\right\}.$$

In order that this product may converge, it is clearly necessary that the number of factors in the numerator may be the same as the number of factors in the denominator; for otherwise the general term of the product would not tend to the value unity as n tends to infinity.

We have therefore $k = l$, and can write (denoting the product by P)

$$P = \prod_{n=1}^{\infty} \left\{ \frac{(n - a_1) \dots (n - a_k)}{(n - b_1) \dots (n - b_k)} \right\}.$$

For large values of n, this general term can be expanded in the form

$$\left(1 - \frac{a_1}{n}\right) \dots \left(1 - \frac{a_k}{n}\right) \left(1 - \frac{b_1}{n}\right)^{-1} \dots \left(1 - \frac{b_k}{n}\right)^{-1},$$

or $$1 - \frac{a_1 + a_2 + \dots + a_k - b_1 - \dots - b_k}{n} + \text{terms in } \frac{1}{n^2} + \dots;$$

In order that the infinite product may be absolutely convergent, it is therefore further necessary that

$$a_1 + \dots + a_k - b_1 - \dots - b_k = 0.$$

We can therefore introduce a factor

$$e^{\frac{a_1 + \dots + a_k - b_1 - \dots - b_k}{n}}$$

in the general term of the product, without altering its value; and we thus have

$$P = \prod_{n=1}^{\infty} \left\{ \frac{\left(1 - \frac{a_1}{n}\right) e^{\frac{a_1}{n}} \left(1 - \frac{a_2}{n}\right) e^{\frac{a_2}{n}} \dots \left(1 - \frac{a_k}{n}\right) e^{\frac{a_k}{n}}}{\left(1 - \frac{b_1}{n}\right) e^{\frac{b_1}{n}} \dots \left(1 - \frac{b_k}{n}\right) e^{\frac{b_k}{n}}} \right\}.$$

But $$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \right\} = \frac{1}{-z\Gamma(-z)\, e^{-\gamma z}}.$$

Therefore $$P = \frac{b_1 \Gamma(-b_1)\, b_2 \Gamma(-b_2) \dots b_k \Gamma(-b_k)}{a_1 \Gamma(-a_1) \dots a_k \Gamma(-a_k)},$$

a formula which expresses the general infinite product P in terms of the Gamma-function.

Example 1. Prove that

$$\prod_{s=1}^{s=\infty} \frac{s(a+b+s)}{(a+s)(b+s)} = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+1)}.$$

Example 2. Shew that

$$x\left(1 - \frac{x}{1^n}\right)\left(1 - \frac{x}{2^n}\right) \dots = \{-\Gamma(-x^{\frac{1}{n}})\,\Gamma(-ax^{\frac{1}{n}}) \dots \Gamma(-a^{n-1}x^{\frac{1}{n}})\}^{-1},$$

where

$$a = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}.$$

97. *Connexion between the Gamma-function and the circular functions.*

We now proceed to establish another of the characteristic properties of the Gamma-function, expressed by the equation

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

We have

$$\Gamma(z)\,\Gamma(1-z) = \frac{1}{z(1-z)} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{z+1-z}}{\left(1+\frac{z}{n}\right)\left(1+\frac{1-z}{n}\right)}$$

$$= \frac{1}{z(1-z)} \prod_{n=1}^{\infty} \frac{n+1}{\left(1+\frac{z}{n}\right)(n+1-z)}$$

$$= \frac{1}{z(1-z)} \prod_{n=1}^{\infty} \frac{1}{\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n+1}\right)}$$

$$= \frac{1}{z} \prod_{n=1}^{\infty} \frac{1}{\left(1-\frac{z^2}{n^2}\right)}$$

$$= \frac{\pi}{\sin \pi z},$$

which is the result stated.

Corollary. If we assign to z the special value $\frac{1}{2}$, this formula gives

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi, \text{ or } \Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}.$$

98. *The multiplication-theorem of Gauss and Legendre.*

We shall next obtain the result

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = \Gamma(nz)\,(2\pi)^{\frac{n-1}{2}}\,n^{\frac{1}{2}-nz}.$$

For let $$\phi(z) = \frac{n^{nz}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right)}{n\Gamma(nz)}.$$

Then $$\phi(z+1) = \frac{n^{nz+n}\,\Gamma(z+1)\,\Gamma\left(z+1+\frac{1}{n}\right)\ldots\Gamma\left(z+1+\frac{n-1}{n}\right)}{n\Gamma(nz+n)}$$

$$= \frac{n^n z\left(z+\frac{1}{n}\right)\ldots\left(z+\frac{n-1}{n}\right)}{(nz+n-1)(nz+n-2)\ldots(nz)}\,\phi(z)$$

$$= \phi(z).$$

It follows from this that $\phi(z)$ is a one-valued function of z, with the period unity; and $\phi(z)$ has no singularities when the real part of z is positive, since $\dfrac{1}{\Gamma(nz)}$ is everywhere regular; it has therefore no singularity for any value of z, and so by Liouville's theorem (§ 47) it is a constant.

Thus $\phi(z)$ is equal to the value which it has when $z=\frac{1}{n}$; which gives

$$\phi(z)=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right).$$

Therefore
$$\overline{\phi(z)}|^2=\left\{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right\}\left\{\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right\}\ldots\left\{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)\right\}$$

(by § 97)
$$=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}}=\frac{(2\pi)^{n-1}}{n}.$$

Thus
$$\phi(z)=\frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}},$$

or
$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right)=\Gamma(nz)\,n^{\frac{1}{2}-nz}(2\pi)^{\frac{n-1}{2}}.$$

Example. If
$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$
shew that
$$B(np,nq)=n^{-nq}\frac{B(p,q)\,B\left(p+\frac{1}{n},q\right)\ldots B\left(p+\frac{n-1}{n},q\right)}{B(q,q)\,B(2q,q)\ldots B\{(n-1)q,q\}}.$$

99. *Expansions for the logarithmic derivates of the Gamma-function.*

We have
$$\{\Gamma(z+1)\}^{-1}=e^{\gamma z}\prod_1^\infty\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

Differentiating logarithmically, this gives

$$\frac{d\log\Gamma(z+1)}{dz}=-\gamma+z\left\{\frac{1}{1(z+1)}+\frac{1}{2(z+2)}+\frac{1}{3(z+3)}+\ldots\right\}.$$

Also
$$\log\Gamma(z+1)=\log z+\log\Gamma(z),$$
so
$$\frac{d}{dz}\log\Gamma(z+1)=\frac{1}{z}+\frac{d}{dz}\log\Gamma(z).$$

Therefore

$$\frac{d^2}{dz^2}\log\Gamma(z) = \frac{1}{z^2} + \frac{d^2}{dz^2}\log\Gamma(z+1)$$

$$= \frac{1}{z^2} + \frac{d}{dz}\left\{\frac{z}{1\,(z+1)} + \frac{z}{2\,(z+2)} + \dots\right\}$$

$$= \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \dots.$$

These expansions are occasionally used in applications of the theory.

100. *Heine's expression of* $\Gamma(z)$ *as a contour integral.*

It has long been recognised that the Gamma-function is intimately connected with the theory of a large and important group of definite integrals; and in fact the function has frequently been defined by means of a definite integral. We now proceed to consider various definite integrals in this connexion, the most general of which is due to Heine and can be obtained in the following way.

We have

$$\Gamma(z) = \frac{1}{z}\prod_{m=1}^{\infty}\frac{\left(1+\frac{1}{m}\right)^z}{1+\frac{z}{m}}.$$

Now if we express $\frac{1}{z}\prod_{m=1}^{n}\frac{m}{z+m}$ in partial fractions, we obtain

$$\frac{1}{z}\prod_{m=1}^{n}\frac{m}{z+m} = \sum_{m=0}^{n}(-1)^m\frac{n!}{m!\,(n-m)!}\frac{1}{z+m}.$$

Consider now the function

$$(-x)^{a-1}.$$

This, when a is a complex quantity, may be defined as being equivalent to

$$e^{(a-1)\log(-x)}.$$

Now the logarithmic function is many-valued, since the value of the function $\log(-x)$ is increased or decreased by $2\pi i$ when the variable x describes a simple circuit round the point $x=0$. In order that the function $(-x)^{a-1}$ may have a unique value, we have therefore to select one of the different determinations of $\log(-x)$: and this may be done in the following way.

We first make the stipulation that the variable x is not to cross the real axis at any point on the positive side of the origin; this prevents x from making circuits round the origin, and so makes each of the determinations of $\log(-x)$ a single-valued function. Then we select, from these determinations, that one which makes $\log(-x)$ real when x is a real negative quantity. The value of $\log(-x)$ being thus uniquely defined for every value of x, it follows that the value of $(-x)^{a-1}$ is likewise uniquely defined.

With these presuppositions, if C be any simple contour enclosing the origin and cutting the real axis in the point $x=1$, we have clearly

$$\int_C (-x)^{a-1}\,dx = \left[_C - \frac{(-x)^a}{a}\right] = -\frac{e^{\pi i a}}{a} + \frac{e^{-\pi i a}}{a} = -\frac{2i\sin\pi a}{a}.$$

Therefore

$$\frac{1}{z}\prod_{m=1}^{n}\frac{\left(1+\frac{1}{m}\right)^z}{1+\frac{z}{m}} = \frac{i}{2\sin\pi z}\left\{\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n}\right)\right\}^z \sum_{m=0}^{n}\int_C (-)^{2m}\binom{n}{m}(-x)^{z+m-1}\,dx$$

$$= \frac{i}{2\sin\pi z}(n+1)^z\int_C (-x)^{z-1}(1-x)^n\,dx.$$

Writing $y=nx$ in this equality, we obtain

$$\frac{1}{z}\prod_{m=1}^{n}\frac{\left(1+\frac{1}{m}\right)^z}{1+\frac{z}{m}} = \frac{i}{2\sin\pi z}\left(1+\frac{1}{n}\right)^z\int_D (-y)^{z-1}\left(1-\frac{y}{n}\right)^n dy,$$

where D denotes any simple contour in the plane of the complex variable y, enclosing the point $y=0$, and cutting the real axis in the point $y=n$. If now we make n increase without limit, we have

$$\Gamma(z) = \frac{i}{2\sin\pi z}\int (-y)^{z-1}e^{-y}\,dy,$$

where the integral is taken along a curve commencing at positive infinity, circulating round the origin in the counter-clockwise direction, and returning to positive infinity again; and in the integrand we must take $(-y)^{z-1}$ as equivalent to $e^{(z-1)\log(-y)}$, where the real value of $\log(-y)$ is to be taken when y is negative, and the logarithm is rendered one-valued by the stipulation that the variable is not to cross the real axis at any point on the positive side of the origin.

Since

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

this result can be written in the form

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int (-y)^{-z}e^{-y}\,dy.$$

This theorem is valid for all values of z—in contrast to that found in the next article, which is true only for restricted values of the variable.

Example 1. *Bourguet's expressions for the Gamma-function.*

By a slight extension of the above proof, it is seen that

$$\Gamma(z) = \frac{1}{2i\sin z\pi}\int y^{z-1}e^{y}\,dy,$$

where the path of integration is restricted only to contain the origin and to be extended indefinitely at both ends in the direction of the negative part of the real axis; the contour need not be closed.

Take then as contour two lines inclined at an angle a to the axis of x, passing through the origin, and a small circle round the origin. The integral round the small circle is zero when z has its real part comprised between 0 and 1. The integration along the two lines gives the result

$$\Gamma(z)=\frac{1}{\sin z\pi}\int_0^\infty \rho^{z-1}e^{\rho\cos a}\sin(\rho\sin a+za)\,d\rho,$$

which can be written in the form

$$\Gamma(z)=\frac{1}{\sin z\pi\,(\sin a)^z}\int_0^\infty \rho^{z-1}e^{\rho\cot a}\sin(\rho+za)\,d\rho.$$

This formula is true for all values of a which are not less than $\frac{\pi}{2}$. Taking a equal to π, we have the result

$$\Gamma(z)=\int_0^\infty \rho^{z-1}e^{-\rho}d\rho.$$

Example 2. By taking for contour of integration a parabola with the origin as focus, shew that

$$\Gamma(z)=\frac{e^{\frac{1}{2}}}{2^{z-1}\sin z\pi}\int_0^\infty e^{-\frac{1}{2}x^2}(1+x^2)^{z-\frac{1}{2}}\cos\left[(2z-1)\tan^{-1}x+x\right]dx. \qquad \text{(Bourguet.)}$$

101. *Expression of* $\Gamma(z)$ *as a definite integral, whose path of integration is real.*

We have, by the result of the preceding article,

$$\Gamma(z)=\frac{i}{2\sin\pi z}\int e^{-y+(z-1)\log(-y)}\,dy.$$

Take a path $ABCDE$, commencing at the positive infinitely distant extremity of the real axis (which considered as initial point we denote by A), proceeding close to the real axis until it arrives at the neighbourhood of the origin, describing a small circle BCD round the origin, and returning, close to the real axis, to positive infinity again (which, considered as terminal point, we denote by E). With the conventions that have been made, the integral along the part AB of the path becomes

$$\frac{i}{2\sin\pi z}\int_\infty^0 e^{-y+(z-1)\log y-i\pi(z-1)}\,dy,$$

in which $\log y$ is supposed to have its real determination.

The part of the integral due to the small circle BCD is easily seen to be zero if the real part of z is positive. For the part of the integral due to DC, we have

$$\frac{i}{2\sin\pi z}\int_0^\infty e^{-y+(z-1)\log y+i\pi(z-1)}\,dy.$$

Thus

$$\Gamma(z) = \frac{i\{e^{i\pi(z-1)} - e^{-i\pi(z-1)}\}}{2\sin\pi z} \int_0^\infty e^{-y+(z-1)\log y}\, dy,$$

or

$$\Gamma(z) = \int_0^\infty e^{-y} y^{z-1}\, dy.$$

This integral is called the *Eulerian Integral of the Second Kind.* It is frequently given as the definition of the Gamma-function: but for this purpose it is unsuited, since the integral exists only when the real part of z is positive.

Example 1. Prove that when z is positive

$$\Gamma(z) = \int_0^1 \left(\log\frac{1}{x}\right)^{z-1} dx.$$

Example 2. Prove that

$$\int_0^\infty e^{-kx} x^{s-1}\, dx = \frac{\Gamma(s)}{k^s}.$$

Example 3. Prove that

$$\frac{1}{(z+1)^s} + \frac{1}{(z+2)^s} + \frac{1}{(z+3)^s} + \dots = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xz} x^{s-1}\, dx}{e^x - 1}.$$

102. *Extension of the definite-integral expression to the case in which the argument of the Gamma-function is negative.*

The formula of the last article is no longer applicable when the argument z is negative. Saalschütz has shewn however that, for negative arguments, an analogous theorem exists. This can be obtained in the following way.

Consider the function

$$\Gamma_1(z) = \int_0^\infty x^{z-1} \left(e^{-x} - 1 + x - \frac{x^2}{2!} + \dots + (-1)^{k+1} \frac{x^k}{k!}\right) dx,$$

where z is a negative number lying between the negative integers $-k$ and $-(k+1)$.

By partial integration we have, when $z < -1$,

$$\Gamma_1(z) = \left[\frac{x^z}{z}\left(e^{-x} - 1 + x - \frac{x^2}{2!} + \dots + (-1)^{k+1}\frac{x^k}{k!}\right)\right]_{x=\infty}$$

$$- \left[\frac{x^z}{z}(-1)^k \left(\frac{x^{k+1}}{(k+1)!} - \frac{x^{k+2}}{(k+2)!} + \frac{x^{k+3}}{(k+3)!} - \dots\right)\right]_{x=0}$$

$$+ \frac{1}{z}\int_0^\infty x^z \left(e^{-x} - 1 + x - \dots + (-1)^k \frac{x^{k-1}}{(k-1)!}\right) dx.$$

The terms in the left-hand member which are not under the integral sign vanish, since $(z+k)$ is negative and $(z+k+1)$ is positive: so we have

$$\Gamma_1(z) = \frac{1}{z}\Gamma_1(z+1).$$

The same proof applies when z lies between 0 and -1, and leads to the result

$$\Gamma(z+1) = z\Gamma_1(z) \qquad (0 > z > -1).$$

The last equation shews that, between the values 0 and -1 of z,

$$\Gamma_1(z) = \Gamma(z).$$

The preceding equation then shews that $\Gamma_1(z)$ is the same as $\Gamma(z)$ for all negative values of z less than -1. Thus for all negative values of z, we have Saalschütz's result

$$\Gamma(z) = \int_0^\infty x^{z-1}\left(e^{-x} - 1 + x - \frac{x^2}{2!} + \ldots + (-1)^{k+1}\frac{x^k}{k!}\right)dx,$$

where k is the integer next less than $-z$.

Example. If a function $P(\mu)$ be such that for positive values of μ we have

$$P(\mu) = \int_0^1 x^{\mu-1}e^{-x}dx,$$

and if for negative values of μ we define $P_1(\mu)$ by the equation

$$P_1(\mu) = \int_0^1 x^{\mu-1}\left(e^{-x} - 1 + x - \ldots + (-1)^{k+1}\frac{x^k}{k!}\right)dx,$$

where k is the integer next less than $-\mu$, shew that

$$P_1(\mu) = P(\mu) - \frac{1}{\mu} + \frac{1}{1!(\mu+1)} - \ldots + (-1)^{k-1}\frac{1}{k!(\mu+k)}. \qquad \text{(Saalschütz.)}$$

103. *Gauss' expression of the logarithmic derivate of the Gamma-function as a definite integral.*

We shall next express the function $\frac{d}{dz}\log\Gamma(z)$ as a definite integral, where z is supposed to be a positive real quantity.

We have $$\frac{1}{s} = \int_0^\infty e^{-sx}dx.$$

Therefore $$\log s = \int_1^s \frac{1}{s}ds = \int_0^\infty \frac{e^{-x} - e^{-sx}}{x}dx.$$

Thus we have

$$\int_0^\infty e^{-s}s^{z-1}\log s\,ds = \int_0^\infty e^{-s}s^{z-1}ds\int_0^\infty \frac{e^{-x}-e^{-sx}}{x}dx$$

$$= \int_0^\infty \frac{dx}{x}\left\{e^{-x}\int_0^\infty e^{-s}s^{z-1}ds - \int_0^\infty e^{-(1+x)s}s^{z-1}ds\right\},$$

or

$$\Gamma'(z) = \Gamma(z)\int_0^\infty \frac{dx}{x}\left\{e^{-x} - \frac{1}{(1+x)^z}\right\}.$$

This equation is due to Dirichlet.

Writing $1+x=e^t$ in the second term of the integral, and $x=t$ in the first term, we have

$$\frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}}\right)dt,$$

which is Gauss' expression of $\frac{d}{dz}\log\Gamma(z)$ as a definite integral.

Example 1. Prove that

$$\frac{d}{dz}\log\Gamma(z) = \int_0^1\left\{\frac{1}{\log\frac{1}{t}} - \frac{t^{z-1}}{1-t}\right\}dt. \qquad \text{(Gauss.)}$$

Example 2. Prove that

$$\frac{d}{dz}\log\Gamma(z) = \log z - \frac{1}{2z} - \int_0^1 da\left[\frac{a(1-a)}{2z(z+1)} + \frac{a(1-a)(2-a)}{3z(z+1)(z+2)} + \dots\right].$$

104. *Binet's expression of $\log\Gamma(z)$ in terms of a definite integral.*

Binet* has given an expression for $\log\Gamma(z)$, which is of great importance as shewing the way in which $\log\Gamma(z)$ increases as z becomes very large; his result will be used later in the derivation of the asymptotic expansion of $\Gamma(z)$.

We have by the last article (z being supposed real and positive)

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}}\right)dt;$$

changing z to $z+1$, we have

$$\frac{d}{dz}\log\Gamma(z+1) = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{e^t-1}\right)dt.$$

Now

$$\int_0^\infty e^{-tz}dt = \frac{1}{z},$$

* *Journal de l'Éc. Polyt.* XVI. (1839), pp. 123—343.

and
$$\int_0^\infty \frac{dt}{t}(e^{-t}-e^{-tz}) = \int_0^\infty dt \int_0^z e^{-ty}\,dy$$
$$= \int_0^z \frac{dy}{y}$$
$$= \log z.$$

Therefore
$$\frac{d}{dz}\log\Gamma(z+1) = \frac{1}{2z} + \log z + \int_0^\infty dt\left\{-\frac{e^{-tz}}{2} - \frac{e^{-t}-e^{-tz}}{t} + \frac{e^{-t}}{t} - \frac{e^{-tz}}{e^t-1}\right\}$$
$$= \frac{1}{2z} + \log z - \int_0^\infty dt\, e^{-tz}\left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right\}.$$

Integrate for z between the limits 1 and z; so
$$\log\Gamma(z+1) = \frac{1}{2}\log z + z(\log z - 1) + \int_0^\infty \left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}\frac{e^{-tz}}{t}\,dt$$
$$+1 - \int_0^\infty \left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}\frac{e^{-t}}{t}\,dt,$$

or
$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \int_0^\infty \left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}\frac{e^{-tz}}{t}\,dt$$
$$+1 - \int_0^\infty \left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}\frac{e^{-t}}{t}\,dt \qquad \text{.................(1)}.$$

The first of these integrals can be otherwise expressed in the following way.

We have*
$$\frac{1}{e^t-1} + \frac{1}{2} - \frac{1}{t} = 2\int_0^\infty \frac{\sin(tu)\,du}{e^{2\pi u}-1}.$$

Multiplying both sides of this equation by $e^{-pt}\,dt$ and integrating with respect to t from zero to infinity, we have
$$\int_0^\infty e^{-pt}\,dt\left(\frac{1}{e^t-1} + \frac{1}{2} - \frac{1}{t}\right) = 2\int_0^\infty e^{-pt}\,dt\int_0^\infty \frac{\sin(tu)\,du}{e^{2\pi u}-1}$$
$$= 2\int_0^\infty \frac{u\,du}{(u^2+p^2)(e^{2\pi u}-1)}.$$

Integrating this equation from $p=z$ to $p=\infty$, we have
$$\int_0^\infty \frac{e^{-zt}}{t}\,dt\left(\frac{1}{e^t-1} + \frac{1}{2} - \frac{1}{t}\right) = 2\int_0^\infty \frac{\tan^{-1}\left(\frac{u}{z}\right)dt}{e^{2\pi u}-1} \qquad \text{...............(2)}.$$

* A proof of this equation can be found by making k infinite in the equation
$$\sum_{n=1}^{k} \frac{2t}{4n^2\pi^2+t^2} = 2\sum_{n=1}^{k}\int_0^\infty e^{-2n\pi u}\sin(tu)\,du.$$

Thus equation (1) becomes

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + 2 \int_0^\infty \frac{\tan^{-1}\left(\frac{u}{z}\right) du}{e^{2\pi u} - 1}$$

$$+ 1 - \int_0^\infty \left\{\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right\} \frac{e^{-t}}{t} \, dt \text{ ....(3).}$$

Now write $z = \frac{1}{2}$ in equation (1): since

$$\Gamma\left(\tfrac{1}{2}\right) = \pi^{\frac{1}{2}},$$

we obtain
$$\frac{1}{2} \log \pi = \frac{1}{2} + \int_0^\infty \left\{\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right\} \frac{e^{-\frac{1}{2}t}}{t} \, dt$$

$$- \int_0^\infty \left\{\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right\} \frac{e^{-t}}{t} \, dt.$$

Write $\frac{t}{2}$ for t in the last integral. Thus

$$0 = \frac{1}{2} \log \pi - \frac{1}{2} - \int_0^\infty \left\{\frac{1}{e^t - 1} - \frac{1}{e^{\frac{1}{2}t} - 1} + \frac{1}{t}\right\} \frac{e^{-\frac{1}{2}t}}{t} \, dt,$$

or
$$0 = \frac{1}{2} \log \pi - \frac{1}{2} - \int_0^\infty \left\{\frac{-1}{e^t - 1} + \frac{e^{-\frac{1}{2}t}}{t}\right\} \frac{dt}{t}.$$

Adding this to equation (3), we obtain

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + 2 \int_0^\infty \frac{\tan^{-1}\left(\frac{u}{z}\right) du}{e^{2\pi u} - 1} + \frac{1}{2} \log \pi + \frac{1}{2}$$

$$+ \int_0^\infty \frac{dt}{t} \left\{\frac{1}{e^t - 1} - \frac{e^{-\frac{1}{2}t}}{t} - \frac{e^{-t}}{e^t - 1} + \frac{e^{-t}}{t} - \frac{e^{-t}}{2}\right\} \text{ ....(4).}$$

The last integral is

$$\int_0^\infty \frac{dt}{t} \left\{\frac{1}{2} e^{-t} + \frac{e^{-t} - e^{-\frac{1}{2}t}}{t}\right\},$$

or
$$\int_0^\infty dt \int_{\frac{1}{2}}^1 \frac{e^{-t} - e^{-xt}}{t} \, dx,$$

or*
$$\int_{\frac{1}{2}}^1 \log x \, dx,$$

* This artifice is due to Pringsheim, *Math. Ann.* xxxi.

or
$$-\tfrac{1}{2}\log\tfrac{1}{2}-\tfrac{1}{2}.$$

Substituting this in equation (4), we obtain

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z - z+\frac{1}{2}\log(2\pi)+2\int_0^\infty \frac{\tan^{-1}\left(\frac{u}{z}\right)du}{e^{2\pi u}-1}.$$

This is Binet's formula for $\log\Gamma(z)$; as z increases indefinitely, the last integral diminishes indefinitely, and so the remaining terms furnish an approximate expression for $\log\Gamma(z)$ when z is large.

Example. Prove that

$$\log\Gamma(z)=(z-\tfrac{1}{2})\log z - z+\tfrac{1}{2}\log(2\pi)+J(z),$$

where $J(z)$ is given by the absolutely convergent series

$$J(z)=\tfrac{1}{2}\left\{\frac{c_1}{z+1}+\frac{c_2}{2(z+1)(z+2)}+\frac{c_3}{3(z+1)(z+2)(z+3)}+\dots\right\},$$

in which

$$c_1=\tfrac{1}{6},\quad c_2=\tfrac{1}{3},\quad c_3=\tfrac{59}{60},\quad c_4=\tfrac{227}{60},$$

and generally

$$c_n=\int_0^1 (x+1)(x+2)\dots(x+n-1)(2x-1)\,x\,dx. \qquad \text{(Binet.)}$$

105. *The Eulerian Integral of the First Kind.*

The name *Eulerian Integral of the First Kind* was given by Binet to the integral

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

which was first studied by Euler and Legendre. In this integral, the real parts of p and q are supposed to be positive; and x^{p-1}, $(1-x)^{q-1}$ are to be understood to mean those values of $e^{(p-1)\log x}$ and $e^{(q-1)\log(1-x)}$ which correspond to the real determinations of the logarithms.

With these stipulations, it is easily seen that the integral exists, since the infinity of the integrand is of less than the first order at the two extremities of the path of integration.

We have, on writing $(1-x)$ for x,

$$B(p,q)=B(q,p).$$

Also

$$\int_0^1 x^{p-1}(1-x)^q\,dx=\left[\frac{x^p(1-x)^q}{p}\right]_0^1+\frac{q}{p}\int x^p(1-x)^{q-1}\,dx,$$

or

$$B(p,q+1)=\frac{q}{p}B(p+1,q).$$

Also

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \int_0^1 (1-x+x)\,x^{p-1}(1-x)^{q-1}\,dx$$

$$= B(p+1, q) + B(p, q+1).$$

Combining these results we obtain the formula

$$B(p, q+1) = \frac{q}{p+q} B(p, q).$$

Example 1. Prove that if n is a positive integer,

$$B(p, n+1) = \frac{1}{p}\,\frac{1 \,.\, 2 \ldots n}{(p+1)\ldots(p+n)}.$$

Example 2. Prove that

$$B(x, y) = \int_0^\infty \frac{a^{x-1}}{(1+a)^{x+y}}\,da.$$

106. *Expression of the Eulerian Integral of the first kind in terms of the Gamma-function.*

We shall now establish the important theorem

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

To prove this, we have

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty e^{-x} x^{m-1}\,dx \times \int_0^\infty e^{-y} y^{n-1}\,dy$$

(writing x^2 for x, and y^2 for y)

$$= 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \times \int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$

(writing $r\cos\theta$ for x, and $r\sin\theta$ for y)

$$= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\,dr\,d\theta$$

$$= \Gamma(m+n)\, 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta$$

(putting $\cos^2\theta = u$) $\qquad = \Gamma(m+n)\,B(m, n).$

This result connects the Eulerian Integral of the first kind with the Gamma-function.

Example. Prove that

$$\int_0^1\int_0^1 f(xy)(1-x)^{\mu-1}y^{\mu}(1-y)^{\nu-1}dx\,dy = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\int_0^1 f(z)(1-z)^{\mu+\nu-1}dz.$$

(Cambridge Mathematical Tripos, Part I., 1894.)

107. *Evaluation of trigonometrical integrals in terms of the Gamma-function.*

We can now evaluate the integral

$$\int_0^{\frac{\pi}{2}}\cos^{m-1}x\sin^{n-1}x\,dx,$$

where m and n are not restricted to be integral, but have their real parts positive.

For writing $\sin^2 x = t$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\cos^{m-1}x\sin^{n-1}x\,dx &= \frac{1}{2}\int_0^1(1-t)^{\frac{m}{2}-1}t^{\frac{n}{2}-1}dt\\ &= \frac{1}{2}B\left(\frac{m}{2},\frac{n}{2}\right)\\ &= \frac{1}{2}\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}.\end{aligned}$$

The well-known elementary formulae for the case in which m and n are integers can be at once derived from this

Example. Prove that when $|k|<1$,

$$\int_0^{\frac{\pi}{2}}\frac{\cos^m\theta\sin^n\theta\,d\theta}{(1-k\sin^2\theta)^{\frac{1}{2}}} = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m+n+1}{2}\right)}\int_0^{\frac{\pi}{2}}\frac{\cos^{m+n}\theta\,d\theta}{(1-k\sin^2\theta)^{\frac{n+1}{2}}}.$$

(Trinity College Examination, 1898.)

108. *Dirichlet's multiple integrals.*

We shall now shew how the integral

$$I = \iiint f\left[\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}+\left(\frac{z}{c}\right)^{\gamma}\right]x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz$$

may be reduced to a simple integral, where f is an arbitrary function of its argument, and the integration is extended over all the systems of positive values of the variables x, y, z, which satisfy the inequality

$$\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}+\left(\frac{z}{c}\right)^{\gamma} \leqslant 1.$$

Write $$x = ax_1^{\frac{1}{\alpha}}, \quad y = by_1^{\frac{1}{\beta}}, \quad z = cz_1^{\frac{1}{\gamma}},$$

$$p = \alpha p_1, \quad q = \beta q_1, \quad r = \gamma r_1.$$

Then the integral takes the form

$$I = \frac{a^p b^q c^r}{\alpha\beta\gamma} \iiint f(x_1 + y_1 + z_1)\, x_1^{p_1-1} y_1^{q_1-1} z_1^{r_1-1}\, dx_1\, dy_1\, dz_1,$$

where the integration is now taken over all the systems of positive values of the variables x_1, y_1, z_1, which satisfy the inequality

$$x_1 + y_1 + z_1 \leqslant 1.$$

Now let
$$f_1 = x_1 + y_1 + z_1 - \xi = 0,$$

$$f_2 = y_1 + z_1 - \xi\eta = 0,$$

$$f_3 = z_1 - \xi\eta\zeta = 0$$

be three equations defining new variables ξ, η, ζ.

Then
$$\frac{\partial(x_1,\ y_1,\ z_1)}{\partial(\xi,\ \eta,\ \zeta)} = \frac{-\dfrac{\partial(f_1,\ f_2,\ f_3)}{\partial(\xi,\ \eta,\ \zeta)}}{\dfrac{\partial(f_1,\ f_2,\ f_3)}{\partial(x_1,\ y_1,\ z_1)}} = -\frac{\begin{vmatrix} -1 & 0 & 0 \\ -\eta & -\xi & 0 \\ -\eta\zeta & -\xi\zeta & -\xi\eta \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \xi^2\eta.$$

The field of integration is clearly such that the new variables ξ, η, ζ each vary from 0 to 1.

Thus

$$I = \frac{a^p b^q c^r}{\alpha\beta\gamma} \int_0^1\int_0^1\int_0^1 f(\xi)\, \xi^{p_1+q_1+r_1-1} (1-\eta)^{p_1-1} \eta^{q_1+r_1-1} (1-\zeta)^{q_1-1} \zeta^{r_1-1}\, d\xi\, d\eta\, d\zeta$$

$$= \frac{a^p b^q c^r}{\alpha\beta\gamma} B(p_1,\ q_1 + r_1)\, B(q_1,\ r_1) \int_0^1 f(\xi)\, \xi^{p_1+q_1+r_1-1}\, d\xi$$

$$= \frac{a^p b^q c^r}{\alpha\beta\gamma} \frac{\Gamma\left(\frac{p}{\alpha}\right) \Gamma\left(\frac{q}{\beta}\right) \Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma}\right)} \int_0^1 f(\xi)\, \xi^{\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} - 1}\, d\xi.$$

The multiple integral is reduced to a simple integral.

It is easily seen that this method of evaluation can be applied to multiple integrals of a similar form in any number of variables.

Example 1. Shew that the moment of inertia of a homogeneous ellipsoid of unit density, taken about the axis of z, is

$$-\tfrac{4}{5}(a^2+b^2)\,\pi abc,$$

where a, b, c are the semi-axes.

Example 2. Shew that the area of the epicycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=l^{\frac{2}{3}}$ is $\frac{3}{8}\pi l^2$.

Note. Dirichlet's integrals can also be evaluated by performing on the variables the substitution

$$x_1=r^2\sin^2\theta_1\sin^2\theta_2,$$

$$y_1=r^2\sin^2\theta_1\cos^2\theta_2,$$

$$z_1=r^2\cos^2\theta_1,$$

leading to the same result as above; in the case of an integral with n variables the corresponding substitution would be

$$x_1=r^2\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{n-1},\text{ etc.}$$

109. *The asymptotic expansion of the logarithm of the Gamma-function (Stirling's series).*

We now proceed to find an expansion which asymptotically (§ 88) represents the function $\log\Gamma(z)$, and is actually used in the calculation of the Gamma-function.

For simplicity, we shall consider only real positive values of the argument z. For a proof and discussion of the expansion when z has complex values the student is referred to a memoir by Stieltjes*.

From Binet's expression for $\log\Gamma(z)$ (§ 104), we have

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+\phi(z),$$

where
$$\phi(z)=2\int_0^\infty\frac{\tan^{-1}\dfrac{x}{z}\,dx}{e^{2\pi x}-1}.$$

Now
$$\tan^{-1}\frac{x}{z}=\frac{x}{z}-\frac{1}{3}\frac{x^3}{z^3}+\frac{1}{5}\frac{x^5}{z^5}-\ldots$$
$$+\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)\,z^{2n-1}}+\frac{(-1)^n}{z^{2n-1}}\int_0^x\frac{t^{2n}dt}{t^2+z^2}.$$

Substituting this in the integral, and remembering that

$$\int_0^\infty\frac{x^{2n-1}\,dx}{e^{2\pi x}-1}=\frac{B_n}{4n},$$

* *Liouville's Journal* (4), v. pp. 425—444 (1889).

where $B_1, B_2, \ldots$ are the Bernoullian numbers, we have

$$\phi(z) = \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r-1)} \frac{1}{z^{2r-1}} + \frac{2(-1)^n}{z^{2r-1}} \int_0^\infty \frac{dx}{e^{2\pi x}-1} \int_0^x \frac{t^{2n}\,dt}{t^2+z^2}.$$

Let us now find approximately the magnitude of the last term when z is very large.

The quantity
$$\int_0^\infty \frac{dx}{e^{2\pi x}-1} \int_0^x \frac{t^{2n}\,dt}{t^2+z^2}$$

is less than

$$\frac{1}{z^2} \int_0^\infty \frac{dx}{e^{2\pi x}-1} \int_0^x t^{2n}\,dt$$

or
$$\frac{1}{(2n+1)z^2} \int_0^\infty \frac{x^{2n+1}\,dx}{e^{2\pi x}-1}$$

or
$$\frac{B_{n+1}}{4(n+1)(2n+1)z^2}.$$

If now any value of n be taken, it is clear that this quantity can be made as small as we please by taking z sufficiently large.

It ollows that the quantity

$$z^{2n-1} \left\{ \phi(z) - \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r-1) z^{2r-1}} \right\}$$

can be made as small as we please by taking a sufficiently large value for z; and therefore (§ 88) the series

$$\frac{B_1}{1.2.z} - \frac{B_2}{3.4.z^3} + \frac{B_3}{5.6.z^5} - \ldots$$

is the asymptotic expansion of the function $\phi(z)$ for large real positive values of z.

We see therefore that the series

$$\left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r(2r-1)} \frac{1}{z^{2r-1}}$$

is the asymptotic expansion of the function $\log \Gamma(z)$ for large real positive values of z. This is generally known as *Stirling's series.*

110. *Asymptotic expansion of the Gamma-function.*

Forming the exponentials of both members of the equation just found, we have

$$\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} e^{\frac{B_1}{1.2.z} - \frac{B_2}{3.4.z^3} + \frac{B_3}{5.6.z^5} - \ldots},$$

or

$$\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left\{1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \dots\right\},$$

where $$C_1 = \frac{B_1}{1 \cdot 2}, \quad C_2 = \frac{B_1^2}{8}, \text{ etc.}$$

Substituting the numerical values of the Bernoullian numbers, the formula becomes

$$\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left\{1 + \frac{1}{12z} + \frac{1}{2(12z)^2} - \frac{139}{30(12z)^3} - \frac{571}{120(12z)^4} - \dots\right\}.$$

This is the *asymptotic expansion of the Gamma-function.* In conjunction with the formula $\Gamma(1+z) = z\Gamma(z)$, it is very useful for the purpose of computing the numerical value of the function.

Tables of the function $\log \Gamma(z)$, correct to 12 decimal places, for values of z between 1 and 2, were constructed in this way by Legendre, and published in his *Exercices de Calcul Intégral*, Tome II. p. 85, in 1817.

Miscellaneous Examples.

1. Shew that

$$(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\dots = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(1+\frac{z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)}.$$

(Trinity College Examination, 1897.)

2. If σ_n be the sum of the n first terms of a divergent series

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots,$$

shew that the series

$$\frac{1}{a_1\sigma_1} + \frac{1}{a_2\sigma_2} + \frac{1}{a_3\sigma_3} + \dots$$

is divergent.

If the squares of the terms of the latter series form a convergent series, shew that a function $G(1+z)$ can be defined by the equation

$$G(1+z) = \text{Limit} \frac{\sigma_n^z}{\left(1+\frac{z}{a_1\sigma_1}\right)\left(1+\frac{z}{a_2\sigma_2}\right)\dots\left(1+\frac{z}{a_n\sigma_n}\right)},$$

and shew that

$$\frac{1}{G(1+z)} = e^{cz} \prod_{n=1}^{\infty} \left\{\left(1+\frac{z}{a_n\sigma_n}\right) e^{-\frac{z}{a_n\sigma_n}}\right\},$$

where c is a constant. (Cesaro.)

3. Prove that

$$\frac{d\log\Gamma(z)}{dz}=\int_0^\infty \frac{e^{-a}-e^{-za}}{1-e^{-a}}\,da-\gamma$$

$$=\int_0^\infty \{(1+a)^{-1}-(1+a)^{-z}\}\frac{da}{a}-\gamma,$$

where γ is Euler's constant.

4. Prove that

$$\Gamma(z)=\operatorname*{Limit}_{n=\infty} n^z B(z, n).$$

5. Prove that

$$B(t+r,\ t-r)=\frac{1}{4^{t-1}}\int_0^\infty \frac{\cosh(2ru)\,du}{\cosh^{2t}u}.$$

6. Prove that, when $q>1$,

$$B(p, q)+B(p+1, q)+B(p+2, q)+\ldots=B(p, q-1).$$

7. Prove that

$$\frac{B(p-a, q)}{B(p, q)}=1+\frac{aq}{p+q}+\frac{a(a+1)q(q+1)}{1.2.(p+q)(p+q+1)}+\ldots.$$

8. Prove that

$$B(p, q)B(p+q, r)=B(q, r)B(q+r, p). \qquad \text{(Euler.)}$$

9. Prove that

$$\log\Gamma(z)=(1-z)\log\pi+\gamma(\tfrac{1}{2}-z)-\tfrac{1}{2}\log\sin z\pi+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log 2n}{n}\sin 2nz\pi.$$

(Kummer.)

10. Prove that

$$\int_0^{\frac{\pi}{2}}\cos^{p+q-2}u\cos(p-q)u\,du=\frac{\pi}{(p+q-1)\,2^{p+q-1}B(p, q)}. \qquad \text{(Cauchy.)}$$

11. Prove that

$$\log B(p, q)=\log\left(\frac{p+q}{pq}\right)+\int_0^1\frac{(1-v^p)(1-v^q)}{(1-v)\log v}\,dv. \qquad \text{(Euler.)}$$

12. Prove that

$$B(p, p+s)=\frac{B(p, p)}{2^s}\left\{1+\frac{s(s-1)}{2(2p+1)}+\frac{s(s-1)(s-2)(s-3)}{2.4.(2p+1)(2p+3)}+\ldots\right\}. \qquad \text{(Binet.)}$$

13. Prove that

$$\frac{\Gamma(p)}{\Gamma(p+\frac{1}{2})}=\left\{\frac{1}{p}+\frac{1}{4p(p+1)}+\frac{1.3^2}{2.4^2.p(p+1)(p+2)}+\ldots\right\}^{\frac{1}{2}}.$$

14. Prove that

$$\left\{\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}\right\}^2=\frac{2p-1}{2}\left\{1+\frac{1}{2(2p+1)}+\frac{1.3^2}{2.4.(2p+3)(2p+5)}+\ldots\right\}.$$

15. Prove that

$$\Gamma(2p)=\frac{2^{2p-1}}{\sqrt{\pi}}\{\Gamma(p)\}^2\left[\frac{2p^2}{2p+1}\left\{1+\frac{1}{2(2p+3)}+\frac{1.3^2}{2.4.(2p+3)(2p+5)}+\ldots\right\}\right]^{\frac{1}{2}}.$$

(Binet.)

16. Prove that

$$\frac{d}{dz}\log\Gamma(z)=-\gamma+\int_0^1\frac{x^{z-1}-1}{x-1}\,dx,$$

where γ is Euler's constant. (Legendre.)

17. Prove that

$$B(p,p)\,B(p+\tfrac{1}{2},p+\tfrac{1}{2})=\frac{\pi}{2^{4p-1}p}.$$ (Binet.)

18. If

$$\int_x^{x+1}\log\Gamma(z)\,dz=u,$$

shew that

$$\frac{du}{dx}=\log x,$$

and hence (or otherwise) that

$$u=x\log x-x+\tfrac{1}{2}\log 2\pi.$$ (Raabe.)

19. Prove that, for all values of z except negative real values,

$$\log\Gamma(z)=(z-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log(2\pi)+\sum_{n=1}^{\infty}\int_0^{\infty}\frac{dx}{x+z}\,\frac{\sin 2n\pi x}{n\pi}.$$

(Bourguet.)

20. Prove that

$$\log\frac{\Gamma(a+1)\,\Gamma(a+b+c+1)}{\Gamma(a+b+1)\,\Gamma(a+c+1)}=\int_0^1\frac{x^a(1-x^b)(1-x^c)}{(1-x)\log\frac{1}{x}}\,dx.$$

21. Prove that

$$\int_0^1\frac{x^{\alpha-1}-x^{\beta-1}}{(1+x)\log x}=\log\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}.$$ (Kummer.)

22. Prove that

$$\log\frac{\Gamma(a+b+1)\,\Gamma(a+c+1)\,\Gamma(b+c+1)}{\Gamma(a+1)\,\Gamma(b+1)\,\Gamma(c+1)\,\Gamma(a+b+c+1)}=\int_0^1\frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)\log\frac{1}{x}}\,dx.$$

23. When x is positive, shew that

$$\sqrt{\pi}\,\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}=\sum_{n=0}^{\infty}\frac{2n\,!}{2^{2n}\,.\,n\,!\;n\,!}\,\frac{1}{x+n}.$$

(Cambridge Mathematical Tripos, Part I, 1897.)

24. If a is positive, shew that

$$\frac{\Gamma(z)\,\Gamma(a+1)}{\Gamma(z+a)}=\sum_{n=0}^{\infty}\frac{(-1)^n a(a-1)(a-2)\ldots(a-n)}{n\,!}\,\frac{1}{z+n}.$$

25. Shew that

$$\int_0^{\frac{\pi}{2}} \cos^p \phi \cos q\phi \, d\phi = 2^{-p-1}\pi \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+q}{2}+1\right)\Gamma\left(\frac{p-q}{2}+1\right)}.$$

26. The curve $r^m = 2^{m-1}\cos m\theta$ is composed of m equal closed loops. Shew that the length of the arc of half of one of the loops is

$$2^{m-1} \cdot \frac{1}{m} \cdot \int_0^{\frac{\pi}{2}} (\cos x)^{\frac{1}{m}-1}\, dx,$$

and hence that the total perimeter of the curve is

$$\frac{\left\{\Gamma\left(\frac{1}{2m}\right)\right\}^2}{\Gamma\left(\frac{1}{m}\right)}.$$

27. Prove that

$$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi)$$
$$+\tfrac{1}{2}\left\{\frac{1}{2\,.\,3}\sum_{r=1}^{\infty}\frac{1}{(z+r)^2} + \frac{2}{3\,.\,4}\sum_{r=1}^{\infty}\frac{1}{(z+r)^3} + \frac{3}{4\,.\,5}\sum_{r=1}^{\infty}\frac{1}{(z+r)^4} + \dots\right\}.$$

28. Prove that

$$\frac{d}{dz}\log\Gamma(z) = \log z - \int_0^1 \frac{x^{z-1}\,dx}{(1-x)\log x}\{\log x + 1 - x\}.$$

29. Prove that

$$\frac{d^2}{dz^2}\log\Gamma(z) = \int_0^\infty \frac{x e^{-xz}\,dx}{1-e^{-x}}.$$

30. Prove that

$$\log\Gamma(z+a) = \log\Gamma(z) + a\log z - \frac{a-a^2}{2z}$$
$$-\frac{a\int_0^1 a(1-a)\,da - \int_0^a a(1-a)\,da}{2z(z+1)}$$
$$-\frac{a\int_0^1 a(1-a)(2-a)\,da - \int_0^a a(1-a)(2-a)\,da}{3z(z+1)(z+2)}$$
$$-\dots\dots$$

31. Prove that

$$\log\Gamma(z) = \int_0^1 \left\{\frac{x^z - x}{x-1} - x(z-1)\right\}\frac{dx}{x\log x}. \qquad \text{(Binet.)}$$

32. Prove that

$$B(p, q) = \frac{p^{p-\frac{1}{2}} q^{q-\frac{1}{2}}}{(p+q)^{p+q-\frac{1}{2}}}(2\pi)^{\frac{1}{2}} e^{M(p,q)},$$

where

$$M(p, q) = 2\rho \int_0^\infty \frac{dt}{e^{2\pi t\rho} - 1} \tan^{-1} \left\{ \frac{(t^3 + t)\rho^3}{pq(p+q)} \right\},$$

and

$$\rho^2 = p^2 + q^2 + pq.$$

33. Expand

$$\{\Gamma(a)\}^{-1}$$

as a series of ascending powers of a.

(Various evaluations of the coefficients in this expansion have been given by Bourguet, *Bull. des Sci. Math.* v. (1881), p. 43; Bourguet, *Acta Math.* II. (1883), p. 261; Schlömilch, *Zeitschrift für Math.* XXV. (1880), pp. 35, 351.)

34. Shew that

$$\int_0^\infty x^m e^{-ax} \cos bx\, dx = \cos\{(m+1)(\pi - \phi)\} \frac{\Gamma(m+1)}{r^{m+1}},$$

and

$$\int_0^\infty x^m e^{-ax} \sin bx\, dx = \sin\{(m+1)(\pi - \phi)\} \frac{\Gamma(m+1)}{r^{m+1}},$$

where

$$-a + bi = r(\cos\phi + i\sin\phi).$$

35. If

$$P(x) = \int_0^1 e^{-z} z^{x-1}\, dz,$$

shew that

$$P(x) = \frac{1}{x} - \frac{1}{1!}\frac{1}{x+1} + \frac{1}{2!}\frac{1}{x+2} - \frac{1}{3!}\frac{1}{x+3} + \dots,$$

and

$$P(x+1) = xP(x) - \frac{1}{e}.$$

36. Prove that

$$\frac{d}{dz} \log \frac{\Gamma(z+x)}{\Gamma(z)} = \frac{x}{z} - \frac{1}{2}\frac{x(x-1)}{z(z+1)} + \frac{1}{3}\frac{x(x-1)(x-2)}{z(z+1)(z+2)} + \dots.$$

37. If a is negative, and if

$$a = -\nu + a,$$

where ν is an integer and a is positive, shew that

$$\frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} = \sum_{n=1}^{\infty} \left\{ \frac{R_n}{x+n} + G_n(x) \right\},$$

where

$$R_n = \frac{(-1)^n (a-1)(a-2)\dots(a-n)}{n!} G(-n),$$

$$G(x) = \left(1 + \frac{x}{a-1}\right)\left(1 + \frac{x}{a-2}\right) \dots\dots \left(1 + \frac{x}{a-\nu}\right),$$

$$G_n(x) = \frac{G(x) - G(-n)}{x+n}.$$

(Hermite.)

38. When $-\infty < a < 1$, shew that

$$\frac{\Gamma(x)\Gamma(a-x)}{\Gamma(a)} = \sum_{n=1}^{\infty} \frac{R_n}{x+n} - \sum_{n=1}^{\infty} \frac{R_n}{x-a-n},$$

where

$$R_n = \frac{(-1)^n a(a+1)\dots(a+n-1)}{n!}.$$

39. When $a > 1$, and ν and α are respectively the integral and fractional parts of a, shew that

$$\frac{\Gamma(x)\Gamma(a-x)}{\Gamma(a)} = \sum_{n=1}^{\infty} \frac{G(x)\rho_n}{x+n} - \sum_{n=1}^{\infty} \frac{G(x)\rho_{\nu+n}}{x-a-n}$$
$$- G(x)\left[\frac{\rho_0}{x-a} + \frac{\rho_1}{x-a-1} + \dots + \frac{\rho_{\nu-1}}{x-a-\nu+1}\right],$$

where

$$G(x) = \left(1-\frac{x}{\alpha}\right)\left(1-\frac{x}{\alpha+1}\right)\dots\left(1-\frac{x}{\alpha+\nu-1}\right)$$

and

$$\rho_n = \frac{(-1)^n \alpha(\alpha+1)\dots(\alpha+n-1)}{n!}.$$

40. If $\rho_1, \rho_2, \dots \rho_\nu$ are the roots of the equation

$$\rho^\nu + a_1\rho^{\nu-1} + \dots + a_\nu = 0,$$

shew that

$$\prod_{n=1}^{\infty} \left\{\left(1 + a_1\frac{x}{z+n} + a_2\frac{x^2}{(z+n)^2} + \dots + a_\nu\frac{x^\nu}{(z+n)^\nu}\right) e^{-\frac{a_1 x}{n}}\right\}$$
$$= \frac{e^{\Gamma'(1) a_1 x}\,\Gamma^\nu(z)}{\Gamma(z-\rho_1 x)\,\Gamma(z-\rho_2 x)\dots\Gamma(z-\rho_\nu x)}.$$

41. If a and b are real and positive, prove that

$$\int_0^\infty \int_0^\infty e^{-\frac{v}{u}-u} v^{b-1} u^{a-b-1}\,du\,dv = \Gamma(a)\,\Gamma(b).$$

42. By taking as contour of integration a parabola with its vertex at the origin, derive from the formula

$$\Gamma(a) = \frac{1}{2i\sin a\pi}\int e^z z^{a-1}\,dz$$

the result

$$\Gamma(a) = \frac{1}{2\sin a\pi}\int_0^\infty e^{-x^2} x^{a-1}(1+x^2)^{\frac{a}{2}}\cdot[3\sin\{x + a\cot^{-1}(-x)\}$$
$$+ \sin\{x + (a-2)\cot^{-1}(-x)\}]\,dx.$$

(Bourguet.)

43. Prove that, when $1 < z < 2$,

$$\int_0^\infty \frac{\sin bx}{x^z}\,dx = \frac{b^{z-1}}{2\Gamma(z)} \frac{\pi}{\sin\frac{\pi z}{2}},$$

and when $0 < z < 1$,

$$\int_0^\infty \frac{\cos bx}{x^z}\,dx = \frac{b^{z-1}}{2\Gamma(z)} \frac{\pi}{\cos\frac{\pi z}{2}}.$$

44. Shew that

$$\int_0^{\frac{\pi}{2}} (1-\tfrac{1}{2}\sin^2 x)^{\frac{2n-1}{2}}\, dx = \frac{n!}{2^{n+2}\pi^{\frac{1}{2}}} \sum_{r=0}^{n} \frac{2^{3r}}{2r!\,(n-r)!} \left\{\Gamma\left(\frac{2r+1}{4}\right)\right\}^2.$$

45. If

$$U = \frac{2^{\frac{x}{2}}}{\Gamma\left(1-\frac{x}{2}\right)}$$

and

$$V = \frac{2^{\frac{x}{2}}}{\Gamma\left(\frac{1}{2}-\frac{x}{2}\right)},$$

and if a function $F(x)$ be defined by the equation

$$F(x) = \sqrt{\pi}\left(V\frac{dU}{dx} - U\frac{dV}{dx}\right),$$

shew (1) that $F(x)$ satisfies the equation

$$F(x+1) = xF(x) + \frac{1}{\Gamma(1-x)},$$

(2) that for all positive integral values of x,

$$F(x) = \Gamma(x),$$

(3) that $F(x)$ is regular for all finite values of x,

(4) that

$$F(x) = \frac{1}{\Gamma(1-x)}\frac{d}{dx}\log\frac{\Gamma\left(\frac{1-x}{2}\right)}{\Gamma\left(1-\frac{x}{2}\right)}.$$

46. Prove that the function $G(x)$, defined by the equation

$$G(x+1) = (2\pi)^{\frac{x}{2}} e^{-\frac{x(x+1)}{2} - \frac{\gamma x^2}{2}} \prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)^k e^{\frac{x^2}{2k}-x},$$

has the properties expressed by the equations

$$G(x+1) = \Gamma(x)\,G(x),$$

$$G(1) = 1,$$

$$\log\frac{G(1-x)}{G(1+x)} = \int_0^x \pi x\cot\pi x\,dx - x\log 2\pi,$$

$$G(x) = \underset{n=\infty}{\text{Limit}}\left[(n+1)^{\frac{(x-1)(x-2)}{2}}\{\Gamma(n+1)\}^{x-1}\prod_{k=0}^{n-1}\frac{\Gamma(1+k)}{\Gamma(x+k)}\right].$$

(Alexerewsky.)

47. If s is a positive quantity (not necessarily integral), and z is a real quantity between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, shew that

$$\cos^s z = \frac{1}{2^{s-1}}\frac{\Gamma(s+1)}{\left\{\Gamma\left(\frac{s}{2}+1\right)\right\}^2}\left\{\tfrac{1}{2} + \frac{s}{s+2}\cos 2z + \frac{s(s-2)}{(s+2)(s+4)}\cos 4z + \dots\right\},$$

and draw graphs of the series and of the function $\cos^s z$.

48. Obtain the expansion

$$\cos^s x = \frac{a}{2^{s-1}} \Gamma(s+1) \left[\frac{\cos ax}{\Gamma\left(\frac{s+a}{2}+1\right)\Gamma\left(\frac{s-a}{2}+1\right)} + \frac{\cos 3ax}{\Gamma\left(\frac{s+3a}{2}+1\right)\Gamma\left(\frac{s-3a}{2}+1\right)} + \dots \right],$$

and find the values of x for which it is applicable.

(Cauchy.)

49. Prove that

$$\int_0^\infty \frac{e^{-x^2}dx}{\left(1+\frac{w^2}{4x^2}\right)^{\frac{s}{2}}} = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}s)} \int_0^\infty e^{-x^2-wx} x^{s-1}\, dx.$$

50. If

$$\zeta(s, x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s},$$

where $|x|<1$ and the real part of x is positive, shew that

$$\zeta(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{xz^{s-1}dz}{e^z - x}$$

and

$$\operatorname*{Limit}_{x=1} (1-x)^{1-s} \zeta(s, x) = \Gamma(1-s).$$

51. If x, w, and s be real, and $0 < w < 1$, and $s > 1$, and if

$$\phi(w, x, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(w+n)^s},$$

shew that

$$\phi(w, x, s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-wz} z^{s-1} dz}{1 - e^{2\pi i x - z}}$$

and

$$\phi(w, x, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ \begin{array}{l} e^{\pi i (\frac{1}{2}s - 2wx)} \phi(x, -w, s) \\ + e^{\pi i \{-\frac{1}{2}s + 2w(1-x)\}} \phi(1-x, w, s) \end{array} \right\}. \quad \text{(Lerch.)}$$

52. If

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

shew that

(1) $$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}dx}{e^x - 1},$$

(2) $$\zeta(s) = (2\pi)^{s-1} \sin \frac{s\pi}{2} \int_0^\infty \left(\frac{e^x+1}{e^x-1} - \frac{2}{x} \right) x^{-s} dx,$$

(3) $$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{\frac{s-1}{2}} \zeta(1-s).$$

53. Let the function $\phi^{(s)}(x)$ be defined by the equation

$$(-1)^s \phi^{(s)}(x) = \int_0^\infty \frac{\chi(t)\, t^s e^{-xt} dt}{\prod_{\nu=1}^{\nu=m} (1 - e^{-a_\nu t})},$$

where z is an integer $\geqslant m$, the function $\chi(t)$ is defined by the equation

$$\chi(t) = \sum_{n=0}^{\infty} \frac{k_n t^n}{n!},$$

and the quantities a_ν are constants whose real part is positive.

Shew that $\phi^{(s)}(x)$ can be expressed by the series

$$\phi^{(s)}(x) = \Sigma f^{(s)}(x+w)\,;$$

where

$$w = \Sigma \lambda_\nu a_\nu,$$

and where

$$(-1)^s f^{(s)}(x) = \int_0^\infty \chi(t)\, t^s e^{-xt}\, dt.$$

Shew also that $\phi^{(s)}(x)$ satisfies the functional equation

$$\phi^{(s)} x - \sum_\nu \phi^{(s)}(x+a_\nu) + \sum_{\mu,\nu} \phi^{(s)}(x+a_\mu+a_\nu) - \ldots + (-1)^m \phi^{(s)}(x+a_1+a_2+\ldots+a_m) = f^{(s)}(x).$$

Shew further that when $\chi(t)=1$, $\phi^{(s)}(x)$ becomes a function $\psi^{(s)}(x)$, which has the multiplication-theorem

$$\psi^{(s)}(nx) = \frac{1}{n^{s+1}} \Sigma \psi^{(s)}\left(x + \frac{\lambda_1 a_1}{n} + \frac{\lambda_2 a_2}{n} + \ldots + \frac{\lambda_m a_m}{n}\right),$$

where all the quantities λ vary from 0 to $(n-1)$.

(Pincherle.)

54. If

$$f_n(x, y, v) = 1 - \binom{n}{1}\frac{x(y+v+n-1)}{y(x+v)} + \binom{n}{2}\frac{x(x+1)(y+v+n-1)(y+v+n)}{y(y+1)(x+v)(x+v+1)} + \ldots,$$

where

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!},$$

shew that

$$f_n(x, y, v) = \frac{\Gamma(y)\,\Gamma(y-x+n)\,\Gamma(x+v)\,\Gamma(v+n)}{\Gamma(y-x)\,\Gamma(y+n)\,\Gamma(v)\,\Gamma(x+v+n)},$$

and that

$$f_n(x, y, v) = \frac{(y-x+n-1)(x+v+n)}{(y-x-1)(x+v)} f_n(x+1, y, v)$$

$$+ \frac{\Gamma(y)\,\Gamma(x+v)}{(y-x-1)\,\Gamma(-n)\,\Gamma(x+1)\,\Gamma(y+v+n-1)}.$$

(Saalschütz.)

CHAPTER X.

LEGENDRE FUNCTIONS.

111. *Definition of Legendre polynomials.*

The expression $$(1-2zh+h^2)^{-\frac{1}{2}}$$

can, when $|h|$ is sufficiently small, be expanded by the multinomial theorem as a series of ascending powers of h, in the form

$$1+hP_1(z)+h^2P_2(z)+h^3P_3(z)+\dots,$$

where
$$P_1(z)=z,$$
$$P_2(z)=\frac{3z^2-1}{2},$$
$$P_3(z)=\frac{5z^3-3z}{2}, \text{ etc.}$$

The expressions $P_1(z)$, $P_2(z)\dots$, which are clearly all polynomials in z, are known as *Legendre polynomials*. $P_n(z)$ is called the *Legendre polynomial of order n.*

It will appear later (§ 116) that these polynomials are particular cases of a more extended class of functions, known as *Legendre functions.*

Example 1. Prove that

$$P_n(\cos\theta)=\frac{(-1)^n}{n!}\operatorname{cosec}^{n+1}\theta\,\frac{d^n(\sin\theta)}{d(\cot\theta)^n}.$$

(Cambridge Mathematical Tripos, Part II, 1893.)

Let θ' be an angle such that

$$(1-2h\cos\theta+h^2)^{-\frac{1}{2}}=\frac{\sin\theta'}{\sin\theta}.$$

Then

$$\cot\theta'=\left\{\frac{1}{\sin^2\theta'}-1\right\}^{\frac{1}{2}}$$
$$=\frac{\cos\theta-h}{\sin\theta}$$
$$=\cot\theta-h\operatorname{cosec}\theta.$$

Therefore by Taylor's theorem we have

$$\sin\theta' = \sum_n \frac{(-h\operatorname{cosec}\theta)^n}{n!}\frac{d^n(\sin\theta)}{d(\cot\theta)^n},$$

or

$$(1-2h\cos\theta+h^2)^{-\frac{1}{2}} = \sum_n \frac{(-h)^n \operatorname{cosec}^{n+1}\theta}{n!}\frac{d^n(\sin\theta)}{d(\cot\theta)^n}.$$

Equating coefficients of h^n, we obtain the required result.

Example 2. Shew that

$$P_n(z) = \frac{1}{n!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(1-z^2)t^2}\left(-\frac{d}{dz}\right)^n e^{-z^2t^2}\,dt.$$

For

$$\int_{-\infty}^{\infty} e^{-at^2}\,dt = \left(\frac{\pi}{a}\right)^{\frac{1}{2}}.$$

Therefore

$$(1-2hz+h^2)^{-\frac{1}{2}} = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-(1-2hz+h^2)t^2}\,dt = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-(1-z^2)t^2}e^{-(z-h)^2t^2}\,dt.$$

Thus

$$\sum_{n=0}^{\infty} h^n P_n(z) = \sum \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(1-z^2)t^2}\left(-\frac{d}{dz}\right)^n e^{-z^2t^2}\frac{h^n}{n!}\,dt,$$

whence the result follows.

Example 3. By equating coefficients of powers of h in the expansion

$$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}}} = \left(1+\tfrac{1}{2}he^{i\theta}+\frac{1.3}{2.4}h^2e^{2i\theta}+\dots\right)$$
$$\times\left(1+\tfrac{1}{2}he^{-i\theta}+\frac{1.3}{2.4}h^2e^{-2i\theta}+\dots\right),$$

shew that

$$P_n(\cos\theta) = \frac{1.3\dots(2n-1)}{2.4\dots 2n}\left\{2\cos n\theta + 2\frac{1.n}{1.(2n-1)}\cos(n-2)\theta+\dots\right\}.$$

112. *Schläfli's integral for $P_n(z)$.*

Let h be any quantity which is not greater than the radius of convergence of the series $\sum_{n=0}^{\infty} h^n P_n(z)$.

Then $(1-2zh+h^2)^{-\frac{1}{2}}$ can be expanded as the series

$$1+hP_1(z)+h^2P_2(z)+h^3P_3(z)+\dots.$$

But $(1-2zh+h^2)^{-\frac{1}{2}}$ is the residue, at the pole

$$t = \frac{1}{h}-\frac{(1-2zh+h^2)^{\frac{1}{2}}}{h},$$

of the function $$-2h^{-1}\left\{\left(t-\frac{1}{h}\right)^2-\frac{1-2zh+h^2}{h^2}\right\}^{-1}.$$

Now the last expression has two poles, namely at the points

$$t = \frac{1}{h} - \frac{(1 - 2zh + h^2)^{\frac{1}{2}}}{h}$$

and

$$t = \frac{1}{h} + \frac{(1 - 2zh + h^2)^{\frac{1}{2}}}{h}.$$

When h is very small, the former of these poles is close to the point $t = z$, while the second pole is in the infinitely distant part of the plane. Therefore, if C be a contour in the t-plane, including the point z, the former pole only is contained within C when h is not large, and so we have

$$\sum_{n=0}^{\infty} h^n P_n(z) = \frac{1}{2\pi i} \int_C \frac{-2}{ht^2 - 2t + 2z - h}\, dt$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} h^n \int_C \frac{(t^2 - 1)^n}{2^n (t - z)^{n+1}}\, dt.$$

Equating coefficients of h^n, we have the result

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{1}{2^n} (t^2 - 1)^n (t - z)^{-n-1}\, dt,$$

which is called *Schläfli's integral-formula* for the Legendre polynomials*.

113. *Rodrigues' formula for the Legendre polynomials.*

From Schläfli's integral

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{1}{2^n} \frac{(t^2 - 1)^n}{(t - z)^{n+1}}\, dt$$

we immediately deduce, by the theorem of § 38, the result

$$P_n(z) = \frac{1}{2^n\, n!} \frac{d^n}{dz^n} (z^2 - 1)^n,$$

which is called *Rodrigues' formula.*

114. *Legendre's differential equation.*

We shall now prove that the function $y = P_n(z)$ is a solution of the differential equation

$$(1 - z^2) \frac{d^2 y}{dz^2} - 2z \frac{dy}{dz} + n(n + 1) y = 0,$$

which is called *Legendre's differential equation of order n.*

* Schläfli, *Ueber die beiden Heine'schen Kugelfunctionen;* Bern, 1881.

For on substituting Schläfli's integral, we have

$$(1-z^2)\frac{d^2P_n(z)}{dz^2} - 2z\frac{dP_n(z)}{dz} + n(n+1)P_n(z)$$

$$= \frac{(n+1)}{2\pi i}\int_C \frac{(t^2-1)^n\,dt}{2^n(t-z)^{n+3}}\{-(n+2)(t^2-1) + 2(n+1)t(t-z)\}$$

$$= \frac{(n+1)}{2\pi i \,.\, 2^n}\int_C \frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right\}dt,$$

and this integral is zero, since the function $(t^2-1)^{n+1}(t-z)^{-n-2}$ resumes its original value after describing the contour C. The Legendre polynomial therefore satisfies the differential equation.

The differential equation can clearly be written in the alternative form

$$\frac{d}{dz}\left\{(1-z^2)\frac{dP_n(z)}{dz}\right\} + n(n+1)P_n(z) = 0.$$

115. *The integral-properties of the Legendre polynomials.*

We shall now shew that

$$\int_{-1}^{1} P_m(z)\,P_n(z)\,dz = 0,$$

and that $$\int_{-1}^{1}\{P_n(z)\}^2\,dz = \frac{2}{2n+1},$$

if m and n are positive integers and m is not equal to n.

For since $$\frac{d}{dz}\left\{(1-z^2)\frac{dP_n}{dz}\right\} + n(n+1)P_n = 0,$$

we have $$\int_{-1}^{1}\left[P_m\frac{d}{dz}\left\{(1-z^2)\frac{dP_n}{dz}\right\} - P_n\frac{d}{dz}\left\{(1-z^2)\frac{dP_m}{dz}\right\}\right]dz$$

$$+ (m-n)(m+n-1)\int_{-1}^{1} P_mP_n\,dz = 0.$$

Integrating by parts, this equation gives

$$-(m-n)(m+n-1)\int_{-1}^{1} P_m(z)\,P_n(z)\,dz$$

$$= \left[(1-z^2)\left\{P_m(z)\frac{dP_n(z)}{dz} - P_n(z)\frac{dP_m(z)}{dz}\right\}\right]_{-1}^{1} = 0,$$

which shews that the integral

$$\int_{-1}^{1} P_m(z)\,P_n(z)\,dz$$

has the value zero when m is not equal to n.

To establish the second part of the theorem, let the equation

$$(1-2hz+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(z)$$

be squared, and the resulting equation integrated between the limits -1 and $+1$; using the result already proved in the first part of the theorem, we thus obtain

$$\int_{-1}^{1} \frac{dz}{1-2hz+h^2} = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \{P_n(z)\}^2 dz,$$

or

$$\sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} \{P_n(z)\}^2 dz = \frac{1}{h} \log \frac{1+h}{1-h}.$$

Equating coefficients of h^{2n} in this equality, we have

$$\int_{-1}^{1} \{P_n(z)\}^2 dz = \frac{2}{2n+1},$$

which is the desired result.

Example 1. Prove that, if m is not equal to n,

$$\int_{-1}^{1} \frac{d^2 P_m(z)}{dz^2} \frac{d^2 P_n(z)}{dz^2} dz = \frac{(n-1)\,n\,(n+1)\,(n+2)}{48} \{3m(m+1) - n(n+1) + 6\} \times \{1 + (-1)^{n+m}\}.$$

(Cambridge Mathematical Tripos, Part I, 1897.)

Example 2. Prove that

$$\int_{-1}^{1} \frac{d^r P_m(z)}{dz^r} \frac{d^r P_n(z)}{dz^r} (1-z^2)^r dz = 0 \text{ or } \frac{2\,(n+r)\,!}{(2n+1)\,(n-r)\,!},$$

according as m, n are unequal or equal.

(Cambridge Mathematical Tripos, Part I, 1893.)

116. *Legendre functions.*

Hitherto we have supposed that the index n of $P_n(z)$ is a positive integer; in fact, $P_n(z)$ has not been defined except when n is a positive integer. We shall now see how the definition can be extended so as to furnish a definition of $P_n(z)$, even when n is not integral.

An analogy can be drawn from the theory of the Gamma-function. The expression $z!$ as ordinarily defined (viz. as $z(z-1)(z-2)\ldots 2 \,.\, 1$) has a meaning only for positive integral values of z; but when the Gamma-function has been introduced, $z!$ can be defined to be $\Gamma(z+1)$, and so a function $z!$ will exist for all values of z.

Referring to § 114, we see the differential equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y = 0$$

is satisfied by the expression

$$y = \frac{1}{2\pi i}\int_C \frac{1}{2^n}(t^2-1)^n (t-z)^{-n-1}\,dt,$$

even when n is not a positive integer, provided that C is a contour such that the function

$$\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}$$

resumes its original value after describing C.

Suppose then that n is no longer taken to be a positive integer.

Now the function $(t^2-1)^{n+1}(t-z)^{-n-2}$ has three singularities, namely the points $t=1$, $t=-1$, $t=z$; and it is clear that after describing a small closed contour enclosing the point $t=1$, the function resumes its original value multiplied by $e^{2\pi i(n+1)}$; while after describing a small closed contour enclosing the point $t=z$, the function resumes its original value multiplied by

$$e^{2\pi i(-n-2)}.$$

If therefore C be a simple contour enclosing the points $t=1$ and $t=z$, but not enclosing the point $t=-1$, then the function

$$\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}$$

will after describing C resume its original value multiplied by $e^{-2\pi i}$, i.e. it will resume its original value. Hence *whatever n be, the Legendrian differential equation of order n,*

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y = 0,$$

is satisfied by the expression

$$y = \frac{1}{2\pi i}\int_C \frac{1}{2^n}(t^2-1)^n (t-z)^{-n-1}\,dt,$$

where C is a simple contour in the t-plane enclosing the points $t=1$ and $t=z$, but not enclosing the point $t=-1$.

This expression will be denoted by $P_n(z)$, and will be termed the Legendre function of the first kind and of order n.

We have thus obtained a definition of $P_n(z)$ which is valid even when n is not integral.

The Legendre function is a mere polynomial when n is integral, but is a new transcendental function when n is not integral; just as $\Gamma(z)$ is the polynomial $(z-1)!$ when z is integral, but is a transcendental function when z is not integral.

We shall suppose the many-valued function z^n, which occurs in the defining integral, to have the value 1 when z is equal to 1, and when z is not equal to 1 to have that value which would be obtained by continuation along a rectilinear path from the point 1 to the point z.

117. *The Recurrence-formulae.*

We proceed to establish a group of formulae which connect Legendre functions of different orders.

We have by § 116, for all real or complex values of n,

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{1}{2^n}\frac{(t^2-1)^n}{(t-z)^{n+1}}\,dt$$

$$= \frac{-1}{2\pi i}\int_C \frac{(t^2-1)^n}{n\,.\,2^n}\,d\left\{\frac{1}{(t-z)^n}\right\}.$$

Integrating by parts, we have

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{1}{n\,.\,2^n(t-z)^n}\,d\{(t^2-1)^n\}$$

$$= \frac{1}{2\pi i}\int_C \frac{t(t^2-1)^{n-1}}{2^{n-1}(t-z)^n}\,dt,$$

and hence we have

$$P_n(z) - zP_{n-1}(z) = \frac{1}{2\pi i}\int_C \frac{(t^2-1)^{n-1}}{2^{n-1}(t-z)^{n-1}}\,dt \quad \text{.......................(A).}$$

Differentiating this equality, we obtain

$$\frac{dP_n(z)}{dz} - z\frac{dP_{n-1}(z)}{dz} - P_{n-1}(z) = \frac{n-1}{2\pi i}\int_C \frac{(t^2-1)^{n-1}}{2^{n-1}(t-z)^n}\,dt = (n-1)\,P_{n-1}(z),$$

so

$$\frac{dP_n(z)}{dz} - z\,\frac{dP_{n-1}(z)}{dz} = nP_{n-1}(z) \quad \text{.....................(I).}$$

This is the first of the required formulae.

Next, from the identity

$$\int_C d\left\{\frac{t(t^2-1)^{n-1}}{(t-z)^{n-1}}\right\} = 0,$$

we deduce

$$0 = \int_C \frac{(t^2-1)^{n-1}}{(t-z)^{n-1}}\,dt + \int_C \frac{2t^2(n-1)(t^2-1)^{n-2}}{(t-z)^{n-1}}\,dt - \int_C (n-1)\frac{t(t^2-1)^{n-1}}{(t-z)^n}\,dt,$$

or

$$0 = \int_C \frac{(t^2-1)^{n-1}}{(t-z)^{n-1}}\,dt + \int_C \frac{2\{(t^2-1)+1\}(n-1)(t^2-1)^{n-2}\,dt}{(t-z)^{n-1}}$$

$$-\int_C \frac{(n-1)\{(t-z)+z\}(t^2-1)^{n-1}}{(t-z)^n}\,dt,$$

or

$$0 = n\int_C \frac{(t^2-1)^{n-1}}{(t-z)^{n-1}}\,dt + (n-1)\int_C \frac{2\,(t^2-1)^{n-2}\,dt}{(t-z)^{n-1}} - (n-1)\,z\int_C \frac{(t^2-1)^{n-1}}{(t-z)^n}\,dt,$$

or

$$0 = \frac{n}{2\pi i}\int_C \frac{(t^2-1)^{n-1}}{2^{n-1}(t-z)^{n-1}}\,dt + \frac{n-1}{2\pi i}\int_C \frac{(t^2-1)^{n-2}\,dt}{2^{n-2}(t-z)^{n-1}}$$

$$-\frac{(n-1)\,z}{2\pi i}\int_C \frac{(t^2-1)^{n-1}}{2^{n-1}(t-z)^n}\,dt,$$

or, by formula (A) above and Schläfli's formula,

$$0 = n\,\{P_n(z) - zP_{n-1}(z)\} + (n-1)\,P_{n-2}(z) - (n-1)\,zP_{n-1}(z),$$

or

$$nP_n(z) - (2n-1)\,zP_{n-1}(z) + (n-1)\,P_{n-2}(z) = 0 \quad\text{.........(II)},$$

a relation connecting three Legendre functions of consecutive orders. This is the second of the required formulae.

Other formulae can be deduced from (I) and (II) in the following way:

Differentiating (II), we have

$$n\frac{dP_n(z)}{dz} - (2n-1)\,z\frac{dP_{n-1}(z)}{dz} + (n-1)\frac{dP_{n-2}(z)}{dz} = (2n-1)\,P_{n-1}(z).$$

Substituting for

$$\frac{dP_n(z)}{dz}$$

from (I), we have

$$n\left\{z\frac{dP_{n-1}(z)}{dz} + nP_{n-1}(z)\right\} - (2n-1)\,z\frac{dP_{n-1}(z)}{dz} + (n-1)\frac{dP_{n-2}(z)}{dz}$$

$$= (2n-1)\,P_{n-1}(z),$$

or

$$-(n-1)\,z\frac{dP_{n-1}(z)}{dz} + (n-1)\frac{dP_{n-2}(z)}{dz} = -(n-1)^2\,P_{n-1}(z),$$

or

$$z\frac{dP_{n-1}(z)}{dz} - \frac{dP_{n-2}(z)}{dz} = (n-1)\,P_{n-1}(z).$$

Changing $(n-1)$ to n in this equality, we have

$$z\frac{dP_n(z)}{dz} - \frac{dP_{n-1}(z)}{dz} = nP_n(z) \quad\text{..................(III)}.$$

Next, changing n to $(n+1)$ in (I), we have

$$\frac{dP_{n+1}(z)}{dz} - z\frac{dP_n(z)}{dz} = (n+1)\,P_n(z).$$

Adding this to (III), we have

$$\frac{dP_{n+1}(z)}{dz} - \frac{dP_{n-1}(z)}{dz} = (2n+1)\,P_n(z) \;\ldots\ldots\ldots\ldots\ldots\text{(IV)}.$$

Lastly, combining (I) and (III), we obtain the result

$$(z^2-1)\frac{dP_n(z)}{dz} = nzP_n(z) - nP_{n-1}(z)\ldots\ldots\ldots\ldots\ldots\ldots\text{(V)}.$$

The formulae (I) to (V) are called the *recurrence-formulae.*

The above proof holds whether n is an integer or not, i.e. it is applicable to the general Legendre functions. Another proof which, however, only applies to the case when n is a positive integer (i.e. is only applicable to the Legendre polynomials) is as follows:

Write

$$V = (1-2hz+h^2)^{-\frac{1}{2}}.$$

Then equating coefficients of powers of h in the equality

$$(1-2hz+h^2)\frac{\partial V}{\partial h} = (z-h)\,V,$$

we have

$$nP_n(z) - (2n-1)\,zP_{n-1}(z) + (n-1)\,P_{n-2}(z) = 0,$$

which is the formula (II).

Similarly, equating coefficients of powers of h in the equality

$$h\frac{\partial V}{\partial h} = (z-h)\frac{\partial V}{\partial z},$$

we have

$$z\frac{dP_n(z)}{dz} - \frac{dP_{n-1}(z)}{dz} = nP_n(z),$$

which is the formula (III). The others can be deduced from these.

Example. Shew that, for all values of n,

$$(2n+3)\,P^2_{n+1} - (2n+1)\,P_n^2 = \frac{d}{dz}\{z\,(P_n^2 + P^2_{n+1}) - 2P_nP_{n+1}\}.$$

(Hargreaves.)

For

$$\frac{d}{dz}\{z\,(P_n^2+P^2_{n+1}) - 2P_nP_{n+1}\}$$

$$= P_n^2 + P^2_{n+1} + 2zP_n\frac{dP_n}{dz} + 2zP_{n+1}\frac{dP_{n+1}}{dz} - 2P_n\frac{dP_{n+1}}{dz} - 2P_{n+1}\frac{dP_n}{dz}$$

$$= P_n^2 + P^2_{n+1} + 2P_n\left\{z\frac{dP_n}{dz} - \frac{dP_{n+1}}{dz}\right\} + 2P_{n+1}\left\{z\frac{dP_{n+1}}{dz} - \frac{dP_n}{dz}\right\}$$

$$= P_n^2 + P^2_{n+1} + 2P_n(-n-1)\,P_n + 2P_{n+1}(n+1)\,P_{n+1}$$

(as is seen by using formulae (I) and (III))

$$= (2n+3)\,P^2_{n+1} - (2n+1)\,P_n^2,$$

which is the required result.

118. *Evaluation of the integral-expression for $P_n(z)$, as a power-series.*

When n is a positive integer, we have seen that $P_n(z)$ is a polynomial in z. When n is not a positive integer, however, $P_n(z)$ is not a polynomial; and as $P_n(z)$ is not a regular function of z for all finite values of z unless n is integral, it follows that no power-series exists which represents $P_n(z)$ for all finite values of z, when n is not integral. In order to find a power-series capable of representing $P_n(z)$, we must therefore make some supposition regarding the part of the z-plane on which the point z lies. We shall suppose that z lies within a circle of radius 2, whose centre is the point 1; so that

$$|1-z|<2.$$

As the contour C of § 116 was subject only to the condition of enclosing $t=1$ and $t=z$ without enclosing $t=-1$, it is clear that we can choose it so as to lie entirely within the circle of centre 1 and radius 2 in the t-plane, i.e. to be such that the inequality $|1-t|<2$ is satisfied for all points t on C.

Now write $t-1=(z-1)u$. When t describes the contour C, the point representing the variable u will describe a contour γ on the u-plane; since C encloses the points $t=z$ and $t=1$, γ will enclose the points $u=1$ and $u=0$; and since $|1-t|<2$, we shall have $|u|<\dfrac{2}{|z-1|}$ for all points u on γ.

Then changing the variable of integration from t to u in the integral which represents $P_n(z)$, we have

$$P_n(z)=\frac{1}{2\pi i}\int_\gamma \frac{1}{2^n}u^n\{(z-1)u+2\}^n(u-1)^{-n-1}\,du$$

$$=\frac{1}{2\pi i}\int_\gamma\left(1+\frac{z-1}{2}u\right)^n u^n(u-1)^{-n-1}\,du.$$

Since $|u|<\dfrac{2}{|z-1|}$ we can expand this in the form

$$P_n(z)=\frac{1}{2\pi i}\sum_{r=0}^{\infty}\int_\gamma\left(\frac{z-1}{2}\right)^r u^{r+n}\,\frac{n(n-1)\dots(n-r+1)}{r!}(u-1)^{-n-1}\,du.$$

Now on integrating by parts, we have the result

$$\int_\gamma u^{r+n}(u-1)^{-n-1}\,du=\left[-u^{r+n}\frac{(u-1)^{-n}}{n}\right]_\gamma+\frac{r+n}{n}\int_\gamma u^{r+n-1}(u-1)^{-n}\,du.$$

The first expression on the right-hand side is zero, and so we have

$$\int_\gamma u^{r+n}(u-1)^{-n-1}\,du=\frac{r+n}{n}\int_\gamma u^{r+n-1}(u-1)^{-n-1}(u-1)\,du$$

$$=\frac{r+n}{n}\int_\gamma u^{r+n}(u-1)^{-n-1}\,du-\frac{r+n}{n}\int_\gamma u^{r+n-1}(u-1)^{-n-1}\,du,$$

or

$$\int_\gamma u^{r+n}(u-1)^{-n-1}\,du=\frac{r+n}{r}\int_\gamma u^{r+n-1}(u-1)^{-n-1}\,du.$$

Therefore

$$\int_\gamma u^{r+n}(u-1)^{-n-1}\,du = \frac{r+n}{r}\cdot\frac{r-1+n}{r-1}\cdots\frac{1+n}{1}\int_\gamma u^n(u-1)^{-n-1}\,du.$$

Now transform the integral on the right-hand side, by writing $u=\dfrac{v}{v-1}$.

The integral $\int_\gamma u^n(u-1)^{-n-1}du$ becomes $-\int_\delta \dfrac{v^n dv}{v-1}$, where the integration has now to be taken in the positive sense round a contour δ enclosing the points $v=0$ and $v=\infty$, but not enclosing the point $v=1$. This integral can be replaced by $+\int_{\delta'}\dfrac{v^n dv}{v-1}$, where the integration has to be taken in the positive sense round a contour δ' enclosing the point $v=1$, but not enclosing the points $v=0$, or $v=\infty$ (since the integrand has no singularities in the region between the contours δ and δ'). The contour δ' can now be diminished until it becomes an infinitesimal circle surrounding the point $v=1$. The value of the integral is then $1^n\int\dfrac{dv}{v-1}$, where the integration is taken round this contour; or $2\pi i$, since the many-valued function v^n has been taken to have the meaning 1 at the point $v=1$. We thus have

$$\frac{1}{2\pi i}\int_\gamma u^{r+n}(u-1)^{n-1}\,du = \frac{r+n}{r}\cdot\frac{r-1+n}{r-1}\cdots\frac{1+n}{1},$$

and on substituting this in the expression already found for $P_n(z)$, we obtain

$$P_n(z)=1+\sum_{r=1}^{\infty}\frac{n(n-1)\ldots(n-r+1)}{r!}\frac{(r+n)(r-1+n)\ldots(1+n)}{r!}\left(\frac{z-1}{2}\right)^r,$$

an expansion of $P_n(z)$ as a series of powers of $(z-1)$.

If now, as in § 14, a series of the form

$$1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2c(c+1)}z^2+\ldots$$

(a *hypergeometric series*) be denoted by

$$F(a,\ b,\ c,\ z),$$

then the expansion can be written in the form

$$P_n(z)=F\left(-n,\ n+1,\ 1,\ \frac{1-z}{2}\right).$$

This is the required expression for $P_n(z)$ as an infinite series. It is valid at all points z within the circle whose equation is $|1-z|<2$.

Corollary. Since this series is clearly unaffected when n is changed to $-n-1$, we have

$$P_n(z) = P_{-n-1}(z).$$

Note. When n is a positive integer, the above series terminates and gives the expression of $P_n(z)$ as a polynomial in $\frac{1-z}{2}$.

119. *Laplace's integral-expression for $P_n(z)$.*

We shall next shew that, for all values of n and for certain values of z, the Legendre function $P_n(z)$ can be represented by the integral (called *Laplace's integral*),

$$\frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\, d\phi.$$

When n is not an integer it is necessary to state which of the branches of the many-valued function in the integrand is to be taken: we shall take that branch of the function $\{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n$ which reduces to unity when taken by the process of continuation along a straight path to the point $z=1$. It will appear later that it is immaterial which branch of the two-valued function $(z^2-1)^{\frac{1}{2}}$ is taken.

(A) *Proof applicable only to the Legendre polynomials.*

When n is a positive integer, the result can easily be obtained in the following way. We have

$$\sum_{n=0}^{\infty} h^n P_n(z) = (1-2hz+h^2)^{-\frac{1}{2}}.$$

But
$$(1-2hz+h^2)^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{(1-hz) - h(z^2-1)^{\frac{1}{2}}\cos\phi},$$

as is seen by applying the ordinary formula for the integration of

$$\int \frac{d\phi}{a + b\cos\phi}.$$

Expanding the integrand of the integral in ascending powers of h, we have

$$(1-2hz+h^2)^{-\frac{1}{2}} = \frac{1}{\pi}\sum_{n=0}^{\infty} h^n \int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n d\phi,$$

and on equating coefficients of h^n on the two sides of this equation, the required result is obtained.

As however the theorem is true whether n is an integer or not (i.e. as it is equally true for the Legendre functions and the Legendre polynomials), it is necessary to have a general proof independent of the character of n; this will now be given.

(B) *General proof.*

First, we shall shew that Laplace's integral satisfies Legendre's equation.

For if we write

$$y = \frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi,$$

we have

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y$$

$$= \frac{n}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\{n\sin^2\phi - 1 - z\cos\phi\,(z^2-1)^{-\frac{1}{2}}\}\,d\phi.$$

But

$$\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\sin^2\phi\,d\phi$$

$$= -\left[\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\sin\phi\cos\phi\right]$$

$$+ \int_0^\pi \cos\phi\,\frac{d}{d\phi}\left[\sin\phi\,\{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\right]d\phi$$

$$= \int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\cos^2\phi\,d\phi$$

$$- (n-2)\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-3}\cos\phi\,(z^2-1)^{\frac{1}{2}}\sin^2\phi\,d\phi$$

$$= \int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}d\phi - (n-1)\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\sin^2\phi\,d\phi$$

$$+ (n-2)\,z\int_0^\pi \sin^2\phi\,d\phi\,\{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-3}.$$

Therefore

$$n\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\sin^2\phi\,d\phi$$

$$= \int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\,d\phi + (n-2)\,z\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-3}\sin^2\phi\,d\phi.$$

Thus we have

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y$$

$$= \frac{n}{\pi}(n-2)\,z\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-3}\sin^2\phi\,d\phi$$

$$- \frac{n}{\pi}\,z\,(z^2-1)^{-\frac{1}{2}}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\cos\phi\,d\phi$$

$$= -\frac{n}{\pi}\,z\,(z^2-1)^{-\frac{1}{2}}\int_0^\pi \frac{d}{d\phi}\left[\{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-2}\sin\phi\right]d\phi$$

$$= 0,$$

which shews that Laplace's integral satisfies Legendre's equation, whatever n and z may be.

Now suppose that z is nearly unity, and put $\frac{1-z}{2} = u$. Then the integral becomes

$$\frac{1}{\pi}\int_0^\pi \{1 - 2u + \cos\phi\,(-4u + u^2)^{\frac{1}{2}}\}^n\, d\phi,$$

which for small values of u can be expanded in the form

$$1 + \frac{1}{\pi}\int_0^\pi d\phi \sum_{r=1}^{\infty} \frac{n(n-1)\dots(n-r+1)}{r!}\{-2u + \cos\phi\,(-4u+u^2)^{\frac{1}{2}}\}^r.$$

This is a series of powers of $u^{\frac{1}{2}}$; the first terms (neglecting $u^{\frac{3}{2}}$) are

$$1 + 2inu^{\frac{1}{2}}\frac{1}{\pi}\int_0^\pi \cos\phi\, d\phi - 2nu\frac{1}{\pi}\int_0^\pi \{1 + (n-1)\cos^2\phi\}\, d\phi,$$

or
$$1 - 2nu\frac{n+1}{2},$$

or
$$1 - n(n+1)u.$$

It is clear that odd powers of $u^{\frac{1}{2}}$ can arise only in conjunction with odd powers of $\cos\phi$ in the integrand, and so here vanish when integrated. Laplace's integral can therefore, when u is small, be expanded in ascending powers of u in the form

$$1 - n(n+1)u + a_2u^2 + a_3u^3 + a_4u^4 + \dots .$$

But the coefficients $a_2, a_3, \dots$ can be found by substituting this expression in Legendre's equation, and equating to zero the coefficients of each power of u. We thus find that

$$a_r = (-1)^r\frac{n(n-1)\dots(n-r+1).(1+n)\dots(r-1+n)(r+n)}{r!\,r!},$$

and thus Laplace's integral is equal to

$$F\left(-n,\ n+1,\ 1,\ \frac{1-z}{2}\right),$$

or (§ 118) to
$$P_n(z).$$

We thus have, for all real or complex values of n, the result

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\, d\phi.$$

It must be observed that as the power-series $F\left(-n,\ n+1,\ 1,\ \frac{1-z}{2}\right)$ was used in the proof, this proof is valid only for values of z which satisfy

the inequality $\frac{|1-z|}{2} < 1$. As however $P_n(z)$ is an analytic function of z, the result will be true for a more extended region including this, provided the integral

$$\frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi$$

is an analytic function of z within this more extended region: since if these two expressions are equal for any region however small in which they are analytic functions, they must be always equal so long as they remain analytic functions. But it is easily seen that for the integral

$$\frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi,$$

every point on the imaginary axis in the z-plane is a singularity: and therefore the region in the z-plane for which the equality

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi$$

is established is the region for which the real part of z is positive.

Corollary. Since

$$P_n(z) = P_{-n-1}(z),$$

we have for all values of n, real or complex, the result

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{-n-1}\,d\phi,$$

so long as the real part of z is positive.

Example. If

$$\frac{1}{(1-2h\cos\theta+h^2)^s} = \sum_{i=0}^{\infty} b_i \cos i\theta, \quad \text{and } s<1,$$

shew that

$$b_i = \frac{2}{\pi}\sin(\pi s)\int_0^1 \frac{h^i x^{i+s-1}\,dx}{(1-x)^s(1-xh^2)^s}.$$

(Binet.)

120. *The Mehler-Dirichlet definite integral for $P_n(z)$.*

Another expression for the Legendre function as a definite integral may be obtained in the following way.

For all values of n, we have by the preceding theorem

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z + \cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi.$$

In this integral, replace the variable ϕ by a new variable h, defined by the equation

$$h = z + (z^2-1)^{\frac{1}{2}}\cos\phi$$

so that

$$dh = -(z^2-1)^{\frac{1}{2}} \sin\phi \, d\phi,$$

and

$$i(1-2hz+h^2)^{\frac{1}{2}} = (z^2-1)^{\frac{1}{2}} \sin\phi.$$

We thus have

$$\frac{i\,dh}{(1-2hz+h^2)^{\frac{1}{2}}} = d\phi,$$

and therefore

$$P_n(z) = \frac{i}{\pi}\int_{z-(z^2-1)^{\frac{1}{2}}}^{z+(z^2-1)^{\frac{1}{2}}} h^n (1-2hz+h^2)^{-\frac{1}{2}}\, dh.$$

Now write $z=\cos\theta$. Thus

$$P_n(\cos\theta) = \frac{i}{\pi}\int_{e^{-i\theta}}^{e^{i\theta}} h^n (1-2h\cos\theta+h^2)^{-\frac{1}{2}}\, dh.$$

Writing $h=e^{i\phi}$, this becomes

$$P_n(\cos\theta) = \frac{1}{\pi}\int_{-\theta}^{\theta} \frac{e^{(n+\frac{1}{2})i\phi}\, d\phi}{(2\cos\phi - 2\cos\theta)^{\frac{1}{2}}},$$

or

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^{\theta} \frac{\cos(n+\frac{1}{2})\phi}{\{2(\cos\phi-\cos\theta)\}^{\frac{1}{2}}}\, d\phi.$$

This is known as *Mehler's simplified form of Dirichlet's integral.* The result is valid for all values of n.

Example 1. Prove that, when n is a positive integer,

$$P_n(\cos\theta) = \frac{2}{\pi}\int_{\theta}^{\pi} \frac{\sin(n+\frac{1}{2})\phi\, d\phi}{\{2(\cos\theta-\cos\phi)\}^{\frac{1}{2}}}.$$

For we have

$$\int_0^{\pi} \frac{dw}{a+b-(a-b)\cos w} = \frac{\pi}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}.$$

Put

$$a=(1+h)^2, \quad b=1-2hy+h^2,$$

$$2\xi = (y-1)+(y+1)\cos w.$$

The equation becomes

$$\frac{\pi}{(1-2hy+h^2)^{\frac{1}{2}}} = \int_{-1}^{y} \frac{(1+h)\, d\xi}{(1-2h\xi+h^2)\{y+\xi(y-1)-\xi^2\}^{\frac{1}{2}}}.$$

Writing $\xi=\cos\phi$, $y=\cos\theta$, this gives

$$(1-2h\cos\theta+h^2)^{-\frac{1}{2}} = \frac{1}{\pi}\int_{\theta}^{\pi}(1+h)\sin\phi\,(1-2h\cos\phi+h^2)\,(1+\cos\phi)^{-\frac{1}{2}}(\cos\theta-\cos\phi)^{-\frac{1}{2}}\, d\phi.$$

Equating coefficients of h^n on both sides, we have

$$P_n(\cos\theta) = \frac{1}{\pi}\int_{\theta}^{\pi} \frac{\sin(n+\frac{1}{2})\phi\sin\phi\, d\phi}{\sin\frac{\phi}{2}\cos\frac{\phi}{2}\{2(\cos\theta-\cos\phi)\}^{\frac{1}{2}}},$$

or

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \{2(\cos\theta - \cos\phi)\}^{-\frac{1}{2}} \sin(n+\tfrac{1}{2})\phi d\phi.$$

Example 2. Prove that

$$P_n(\cos\theta) = \frac{1}{2\pi i}\int \frac{h^n}{(h^2 - 2h\cos\theta + 1)^{\frac{1}{2}}} dh,$$

the integral being taken along a closed path which encloses the two points $h = e^{\pm i\theta}$, and the conventional meaning being assigned to the radical.

Hence (or otherwise) prove that, if θ lie between $\frac{1}{6}\pi$ and $\frac{5}{6}\pi$,

$$P_n(\cos\theta) = \frac{4}{\pi}\frac{2.4\ldots 2n}{3.5\ldots(2n+1)}\left\{\begin{array}{l}\dfrac{\cos(n\theta+\phi)}{(2\sin\theta)^{\frac{1}{2}}} + \dfrac{1^2}{2(2n+3)}\dfrac{\cos(n\theta+3\phi)}{(2\sin\theta)^{\frac{3}{2}}} \\ + \dfrac{1^2.3^2}{2.4.(2n+3)(2n+5)}\dfrac{\cos(n\theta+5\phi)}{(2\sin\theta)^{\frac{5}{2}}} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\},$$

where ϕ denotes $\frac{1}{2}\theta - \frac{1}{4}\pi$.

Shew also that the first few terms of the series give an approximate value of $P_n(\cos\theta)$ for all values of θ between 0 and π which are not nearly equal to either 0 or π. And explain how this theorem may be used to approximate to the roots of the equation $P_n(\cos\theta) = 0$.

(Cambridge Mathematical Tripos, Part II, 1895.)

121. *Expansion of $P_n(z)$ as a series of powers of $\frac{1}{z}$.*

We now proceed to find an expansion of the Legendre function which is valid for large values of z.

If the real part of z be positive, we have for all values of n (from Laplace's integral)

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z + (z^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi.$$

Now suppose that $|z|$ is very large: then this can be written in the form

$$P_n(z) = \frac{z^n}{\pi}\int_0^\pi \left\{1 + \left(1 - \frac{1}{z^2}\right)^{\frac{1}{2}}\cos\phi\right\}^n d\phi.$$

Expanding the integrand in ascending powers of $\frac{1}{z}$, this gives

$$P_n(z) = \frac{z^n}{\pi}\int_0^\pi \left\{(1+\cos\phi) - \frac{\cos\phi}{2z^2} + \ldots\right\}^n d\phi$$

$$= \frac{z^n}{\pi}\int_0^\pi \left\{(1+\cos\phi)^n - \frac{n\cos\phi}{2z^2}(1+\cos\phi)^{n-1} + \ldots\right\} d\phi.$$

We can evaluate

$$\int_0^\pi (1+\cos\phi)^n d\phi \text{ and } \int_0^\pi \cos\phi(1+\cos\phi)^{n-1} d\phi$$

by putting $\phi = 2\psi$ and using the result

$$\int_0^{\frac{\pi}{2}} \cos^{2n}\psi d\psi = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)};$$

and thus we find that $P_n(z)$ can be expressed by a series of powers of $\frac{1}{z}$, the first two terms of the expansion being given by the equation

$$P_n(z) = \frac{2^n z^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)} \left\{1 - \frac{n(n-1)}{(2n-1)\,2z^2} + \dots\right\}.$$

The general law of the coefficients in the series can without difficulty be found by substituting in Legendre's differential equation (§ 114); and in this way we find that $P_n(z)$ *can be expressed by the hypergeometric series*

$$P_n(z) = \frac{2^n z^n \Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,.\,\pi^{\frac{1}{2}}} F\left(\frac{1-n}{2}, \ -\frac{n}{2}, \ \frac{1}{2}-n, \ \frac{1}{z^2}\right),$$

in the notation of § 14.

This series has only been proved to hold when z is large and the real part of z is positive: but by § 14 it converges, and so represents an analytical function, over all the area outside the circle of centre O and radius 1. The series therefore represents $P_n(z)$ over this region.

122. *The Legendre functions of the second kind.*

Hitherto we have considered only one solution of the Legendre differential equation, namely $P_n(z)$. We can now proceed to find a second solution.

It appears from § 114, that the differential equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y = 0$$

is satisfied by the integral

$$\int (t^2-1)^n (t-z)^{-n-1}\,dt,$$

taken round any contour such that the integrand resumes its initial value after making the circuit of it. Let D be a figure-of-eight contour in the t-plane, enclosing the point $t = +1$ in one loop and the point $t = -1$ in the other, and not enclosing the point $t = z$. Then after describing this contour, the above integrand clearly resumes its initial value, since it acquires the factor $e^{2\pi i n}$ after describing the first loop, and this is destroyed by the factor $e^{-2\pi i n}$ acquired during the description of the second loop. D is therefore a possible contour.

A solution of Legendre's equation is therefore furnished by the function $Q_n(z)$, if $Q_n(z)$ be defined by the equation

$$Q_n(z) = \frac{1}{4i \sin n\pi} \int_D \frac{1}{2^n} (t^2 - 1)^n (z - t)^{-n-1} \, dt;$$

it is supposed that, in describing D, the point t makes a positive, i.e. counter-clockwise, turn round the point $t = -1$, and then a negative, i.e. clockwise, turn round the point $t = +1$. The significance of the many-valued functions $(t^2 - 1)^n$ and $(t - z)^{-n-1}$ will be supposed to be fixed in the same way as before.

Another form of the integral may be obtained in the following way.

Let the contour become so attenuated as to consist simply of a line joining the points -1 and $+1$, described twice, and two small circles round the points -1 and $+1$: when the real part of $(n + 1)$ is positive, the parts of the integral arising from these two loops are at once seen to be infinitesimal; and thus we have

$$\int_D (t^2 - 1)^n (t - z)^{-n-1} \, dt = (e^{n\pi i} - e^{-n\pi i}) \times \int_{-1}^1 (1 - t^2)^n (t - z)^{-n-1} \, dt$$

$$= 2i \sin n\pi \int_{-1}^1 (1 - t^2)^n (t - z)^{-n-1} \, dt,$$

so
$$Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1 - t^2)^n (z - t)^{-n-1} \, dt.$$

This last result is valid when the real part of $(n + 1)$ is positive. When n is a positive integer, the original definition of $Q_n(z)$ becomes undeterminate: in this case we can use the formulae just found.

$Q_n(z)$ is called *the Legendre function of the second kind and of order n.*

123. *Expansion of $Q_n(z)$ as a power-series.*

We now proceed to express the Legendre function of the second kind as a power-series in $\frac{1}{z}$.

We have, when the real part of $(n + 1)$ is positive,

$$Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1 - t^2)^n (z - t)^{-n-1} \, dt.$$

Suppose that $|z| > 1$. Then the integral can be expanded in the form

$$Q_n(z) = \frac{1}{2^{n+1} z^{n+1}} \int_{-1}^1 (1 - t^2)^n \left(1 - \frac{t}{z}\right)^{-n-1} dt$$

$$= \frac{1}{2^{n+1} z^{n+1}} \int_{-1}^1 (1 - t^2)^n \left\{1 + \sum_{r=1}^{\infty} \left(\frac{t}{z}\right)^r \frac{(n+1)(n+2)\ldots(n+r)}{r!}\right\} dt$$

$$= \frac{1}{2^{n+1} z^{n+1}} \left[\int_{-1}^1 (1 - t^2)^n \, dt + \sum_{s=1}^{\infty} \frac{(n+1)\ldots(n+2s)}{2s!\, z^{2s}} \int_{-1}^1 (1 - t^2)^n t^{2s} \, dt\right],$$

as is seen on writing r for $2s$, since the integrals arising from odd values of r obviously vanish.

Writing $t^2=u$, we can evaluate the coefficients of powers of $\frac{1}{z}$ as follows:

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^n t^{2s}dt &= \int_0^1 (1-u)^n u^{s-\frac{1}{2}}du\\ &= B(n+1, s+\tfrac{1}{2})\\ &= \frac{\Gamma(n+1)\,\Gamma(s+\frac{1}{2})}{\Gamma(n+s+\frac{3}{2})},\end{aligned}$$

and thus the formula for $Q_n(z)$ becomes

$$Q_n(z)=\frac{\pi^{\frac{1}{2}}}{2^{n+1}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\frac{1}{z^{n+1}}F\left(\frac{n+1}{2},\ \frac{n+2}{2},\ n+\frac{3}{2},\ \frac{1}{z^2}\right).$$

This is the expansion of the Legendre function of the second kind as a power-series in $\frac{1}{z}$, corresponding to the expansion obtained for $P_n(z)$ in § 121. The proof given above applies only when the real part of $(n+1)$ is positive; but a similar process can be applied to the integral

$$Q_n(z)=\frac{1}{4i\sin n\pi}\int_D \frac{1}{2^n}(t^2-1)^n(z-t)^{-n-1}\,dt,$$

the coefficients being evaluated in the same way as those which occurred in the expansion of the Legendre function $P_n(z)$ in ascending powers of $\frac{1-z}{2}$; the same result is reached, which shews that the formula

$$Q_n(z)=\frac{\pi^{\frac{1}{2}}}{2^{n+1}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\frac{1}{z^{n+1}}F\left(\frac{n+1}{2},\ \frac{n+2}{2},\ n+\frac{3}{2},\ \frac{1}{z^2}\right)$$

is true for all values of n, real or complex, and for all values of z represented by points outside the circle of centre O and radius unity.

Example 1. Shew that, when n is a positive integer,

$$Q_n(z)=\frac{(-2)^n n!}{2n!}\frac{d^n}{dz^n}\left\{(z^2-1)^n\int_z^\infty (v^2-1)^{-n-1}dv\right\}.$$

We can write Legendre's differential equation in the form

$$(1-z^2)\frac{d^2u}{dz^2}-2z\frac{du}{dz}+n(n+1)u=0.$$

It is easily verified that this equation can be derived from the equation

$$(1-z^2)\frac{d^2x}{dz^2}+2(n-1)z\frac{dx}{dz}+2nx=0,$$

by differentiating n times and writing $u=\frac{d^nx}{dz^n}$.

Now one solution of the latter equation is $x=(z^2-1)^n$; and a second solution can be derived by the ordinary process for finding a second solution of a linear differential equation of the second order, of which one solution is known. Thus two independent solutions of this equation are found to be

$$(z^2-1)^n \text{ and } (z^2-1)^n \int_z^\infty (v^2-1)^{-n-1} dv.$$

It follows that

$$\frac{d^n}{dz^n}\left\{(z^2-1)^n \int_z^\infty (v^2-1)^{-n-1} dv\right\}$$

is a solution of Legendre's equation. As this expression, when expanded in ascending powers of $\frac{1}{z}$, commences with a term in z^{-n-1}, it must be a constant multiple of $Q_n(z)$; and on comparing the coefficient of z^{-n-1} in this expression with the coefficient of z^{-n-1} in the expansion of $Q_n(z)$, as found above, we obtain the required result.

Example 2. Shew that, when n is a positive integer, the Legendre function of the second kind can be expressed by the formula

$$Q_n(z)=2^n n! \int_z^\infty \int_v^\infty \int_v^\infty \dots \int_v^\infty (v^2-1)^{-n-1} (dv)^{n+1}.$$

For on expanding the integrand $(v^2-1)^{-n-1}$ in ascending powers of $\frac{1}{v}$, the right-hand side of the equation takes the form

$$2^n n! \int_z^\infty \int_v^\infty \int_v^\infty \dots \int_v^\infty (dv)^{n+1} \left\{\frac{1}{v^{2n+2}}+\frac{n+1}{v^{2n+4}}+\frac{(n+1)(n+2)}{2!\, v^{2n+6}}+\dots\right\},$$

and on performing the integrations this becomes

$$\frac{n!}{(2n+1)(2n-1)\dots 3.1}\left\{\frac{1}{z^{n+1}}+\frac{(n+1)(n+2)}{2(2n+3)}\frac{1}{z^{n+3}}+\dots\right\},$$

or $Q_n(z)$.

Example 3. Shew that, when n is a positive integer,

$$Q_n(z)=\sum_{t=0}^{n} \frac{2^n . n!}{t!\,(n-t)!}(-z)^{n-t}\int_z^\infty v^t (v^2-1)^{-n-1} dv.$$

This result can be obtained by applying the general integration-theorem

$$\int_z^\infty \int_v^\infty \int_v^\infty \dots \int_v^\infty f(v)\,(dv)^{n+1}=\sum_{t=0}^{n}\frac{(-z)^{n-t}}{t!\,(n-t)!}\int_z^\infty v^t f(v)\,dv$$

to the preceding result.

124. *The recurrence-formulae for the Legendre function of the second kind.*

The functions $P_n(z)$ and $Q_n(z)$ have been defined by integrals of precisely the same form, namely

$$\int (t^2-1)^n (t-z)^{-n-1} dt.$$

It follows therefore that the general proof of the recurrence-formulae for $P_n(z)$, given in § 117, is equally applicable to the function $Q_n(z)$; and hence that *the Legendre function of the second kind satisfies the recurrence-formulae*

$$\frac{dQ_n(z)}{dz} - z\frac{dQ_{n-1}(z)}{dz} = nQ_{n-1}(z),$$

$$nQ_n(z) - (2n-1)zQ_{n-1}(z) + (n-1)Q_{n-2}(z) = 0,$$

$$z\frac{dQ_n(z)}{dz} - \frac{dQ_{n-1}(z)}{dz} = nQ_n(z),$$

$$\frac{dQ_{n+1}(z)}{dz} - \frac{dQ_{n-1}(z)}{dz} = (2n+1)\,Q_n(z),$$

$$(z^2-1)\frac{dQ_n(z)}{dz} = nzQ_n(z) - nQ_{n-1}(z).$$

125. *Laplace's integral for the Legendre function of the second kind.*

Consider the expression

$$y = \int_0^\infty \{z + \cosh\theta\,(z^2-1)^{\frac{1}{2}}\}^{-n-1}\,d\theta,$$

in which z is supposed not to be a real negative number between -1 and $-\infty$, and the real part of $(n+1)$ is supposed to be positive; under these conditions the integral certainly exists.

If now we form the quantity

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)\,y$$

(which occurs in Legendre's differential equation), we find for it the value

$$-(n+1)^2\int_0^\infty \{z + (z^2-1)^{\frac{1}{2}}\cosh\theta\}^{-n-3}\sinh^2\theta\,d\theta$$

$$+(n+1)\int_0^\infty \{z + (z^2-1)^{\frac{1}{2}}\cosh\theta\}^{-n-3}\,d\theta$$

$$+(n+1)\,z\,(z^2-1)^{-\frac{1}{2}}\int_0^\infty \{z + (z^2-1)^{\frac{1}{2}}\cosh\theta\}^{-n-3}\cosh\theta\,d\theta.$$

This expression can be transformed, by integration by parts, in exactly the same manner as the corresponding expression found in the discussion of Laplace's integral for $P_n(z)$, in § 119; and thus it is found to be zero. The quantity y therefore satisfies Legendre's equation.

In order to compare y with the solutions $P_n(z)$ and $Q_n(z)$ which have already been found, we suppose that $|z|$ is large, and write y in the form

$$z^{-n-1}\int^\infty \left\{1 + \cosh\theta\left(1 - \frac{1}{2z^2} + \dots\right)\right\}^{-n-1}d\theta,$$

which when expanded as a power-series becomes

$$\frac{a_0}{z^{n+1}} + \frac{a_1}{z^{n+3}} + \frac{a_2}{z^{n+5}} + \frac{a_3}{z^{n+7}} + \dots \qquad (1),$$

where

$$a_0 = \int_0^\infty (1 + \cosh\theta)^{-n-1} d\theta$$

$$= \int_0^1 2^{-n-1} \cdot \nu^n (1-\nu)^{-\frac{1}{2}} d\nu, \quad \text{where } \nu = \frac{2}{1+\cosh\theta},$$

$$= \frac{1}{2^{n+1}} B(n+1, \tfrac{1}{2}), \quad \text{where } B \text{ is the Eulerian integral of the first kind,}$$

$$= \frac{\pi^{\frac{1}{2}}}{2^{n+1}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}.$$

Now any expression of the form (1) which satisfies Legendre's differential equation must be a multiple of $Q_n(z)$ (since, by substituting the expansion in the differential equation, we can determine the coefficients $a_1, a_2, a_3, \dots$ uniquely in terms of a_0, which shews that all expressions of the kind are multiples of any one of them); and as the value found for a_0 is equal to the coefficient of the initial term in the expansion of $Q_n(z)$, we have

$$y = Q_n(z).$$

Thus we have the result

$$Q_n(z) = \int_0^\infty \{z + (z^2-1)^{\frac{1}{2}} \cosh\theta\}^{-n-1} d\theta,$$

which may be regarded as the analogue of the Laplace's integral already found (§ 119) for $P_n(z)$.

The theorem is valid only when the real part of $(n+1)$ is positive; and the proof has assumed that $|z| > 1$; but the equivalence of $Q_n(z)$ and the integral, having been proved to subsist for this range of values of z, must continue to subsist for all values of z, continuous with this range, for which the integral continues to represent an analytic function of z; and hence the theorem holds for all values of z except those which are real and less than -1, which are singularities of the integral.

126. *Relation between $P_n(z)$ and $Q_n(z)$, when n is integral.*

When n is a positive integer, and z is not a real number between 1 and -1, the functions $Q_n(z)$ and $P_n(z)$ are connected by the relation

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} P_n(y) \frac{dy}{z-y},$$

which we shall now establish.

When $|z| > 1$, we have

$$\frac{1}{2}\int_{-1}^{1} P_n(y)\frac{dy}{z-y} = \frac{1}{2}\int_{-1}^{1} P_n(y)\frac{dy}{z}\left(1+\frac{y}{z}+\frac{y^2}{z^2}+\frac{y^3}{z^3}+\dots\right).$$

Now if $(n+k)$ is an odd integer, we have

$$\int_{-1}^{1} P_n(y)\, y^k dy = \int_0^1 P_n(y)\, y^k dy - \int_0^1 P_n(y)\, y^k dy = 0.$$

If n is less than k, and $(n+k)$ is an even integer, we have

$$\frac{1}{2}\int_{-1}^{1} P_n(y)\, y^k dy = \int_0^1 P_n(y)\, y^k dy$$

(by Rodrigues' theorem) $$= \frac{1}{2^n n!}\int_0^1 y^k \frac{d^n}{dy^n}(y^2-1)^n\, dy$$

(integrating by parts) $$= \frac{1}{2^n n!} k(k-1)\dots(k-n+1)\int_0^1 y^{k-n}(1-y^2)^n\, dy$$

$$= \frac{1}{2^{n+1} n!} k(k-1)(k-2)\dots(k-n+1)\, B\left(\frac{k-n+1}{2},\ n+1\right)$$

$$= \frac{k(k-1)(k-2)\dots(k-n+1)}{(k+n+1)(k+n-1)\dots(k-n+1)}.$$

If on the other hand k is less than n, and $(n+k)$ is an even integer, the same process shews that the integral vanishes.

Therefore

$$\frac{1}{2}\int_{-1}^{1} P_n(y)\frac{dy}{z-y} = \Sigma\, \frac{k(k-1)\dots(k-n+1)}{(k+n+1)(k+n-1)\dots(k-n+1)}\, \frac{1}{z^{k+1}},$$

where the summation is taken for the values $k = n,\ n+2,\ n+4,\ n+6,\ \dots \infty$. But this expansion, by § 123, represents $Q_n(z)$. The theorem is thus established for the case in which $|z| > 1$. Since each side of the equation

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} P_n(y)\frac{dy}{z-y}$$

represents an analytic function even when $|z|$ is not greater than unity, provided z is not a real number between -1 and $+1$, it follows that, with this exception, the result is true universally.

Example. Shew that $Q_n(z)$, where n is a positive integer, is the coefficient of h^n in the expansion of $(1-2hz+h^2)^{-\frac{1}{2}}\cosh^{-1}\left\{\frac{h-z}{(z^2-1)^{\frac{1}{2}}}\right\}$.

For

$$\sum_{n=0}^{\infty} h^n Q_n(z) = \sum_{n=0}^{\infty}\frac{h^n}{2}\int_{-1}^{1}\frac{P_n(y)\,dy}{z-y}$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{(1-2hy+h^2)^{-\frac{1}{2}}dy}{(z-y)}$$

$$= (1-2zh+h^2)^{-\frac{1}{2}}\cosh^{-1}\left\{\frac{h-z}{(z^2-1)^{\frac{1}{2}}}\right\}.$$

127. *Development of the function* $(t-x)^{-1}$ *as a series of Legendre polynomials in* x.

We shall now obtain an expansion which will serve as the basis of a general class of expansions involving Legendre functions.

We have, by the recurrence-formulae,

$$(2n+1)\,xP_n(x)-(n+1)\,P_{n+1}(x)-nP_{n-1}(x)=0,$$

$$(2n+1)\,zP_n(z)-(n+1)\,P_{n+1}(z)-nP_{n-1}(z)=0.$$

Multiply the first of these equations by $P_n(z)$, the second by $P_n(x)$, and subtract: we thus obtain

$$\begin{aligned}(2n+1)(z-x)\,P_n(z)\,P_n(x)&\\ &=(n+1)\{P_{n+1}(z)\,P_n(x)-P_n(z)\,P_{n+1}(x)\}\\ &\qquad -n\{P_n(z)\,P_{n-1}(x)-P_n(x)\,P_{n-1}(z)\}.\end{aligned}$$

Write $n=0,\ 1,\ 2,\ 3,\ldots n$ successively, and add the resulting equations. This gives

$$\begin{aligned}\{P_0(x)\,P_0(z)+3P_1(x)\,P_1(z)+\ldots+(2n+1)\,P_n(x)\,P_n(z)\}\,(z-x)&\\ =(n+1)\{P_{n+1}(z)\,P_n(x)-P_{n+1}(x)\,P_n(z)\}.&\end{aligned}$$

Divide throughout by $(z-x)(z-t)$, and integrate from $z=-1$ to $z=+1$.

Thus

$$\begin{aligned}&\sum_0^n\int_{-1}^{+1}(2r+1)\,P_r(x)\,\frac{P_r(z)}{z-t}\,dz\\ &\qquad=\int_{-1}^{+1}\frac{n+1}{(z-x)(z-t)}\{P_{n+1}(z)\,P_n(x)-P_{n+1}(x)\,P_n(z)\}\,dz\end{aligned}$$

(by partial fractions)
$$\begin{aligned}&=\frac{1}{x-t}\left[\int_{-1}^{+1}\frac{n+1}{z-x}\{P_{n+1}(z)\,P_n(x)-P_{n+1}(x)\,P_n(z)\}\,dz\right.\\ &\qquad\left.-\int_{-1}^{+1}\frac{n+1}{z-t}\{P_{n+1}(z)\,P_n(x)-P_{n+1}(x)\,P_n(z)\}\,dz\right].\end{aligned}$$

Now by the result of the last article, the left-hand side of this equation can be written

$$-2\sum_0^n(2r+1)\,P_r(x)\,Q_r(t).$$

In the first integral on the right-hand side, replace the integrand by its value $\sum\limits_0^n(2r+1)\,P_r(x)\,P_r(z)$, and integrate: only the first term survives, since

$$\int_{-1}^{+1}P_r(z)\,dz=0,$$

when r is an integer greater than zero; so the integral has the value 2.

We thus have

$$\sum_{r=0}^{\infty}(2r+1)\,P_r(x)\,Q_r(t)=\frac{1}{t-x}+\frac{n+1}{t-x}\{P_n(x)\,Q_{n+1}(t)-P_{n+1}(x)\,Q_n(t)\}.$$

This equation is valid for all values of n. Let us now see if x and t can be so chosen as to make the last part of the right-hand side tend to zero as n tends to infinity. We have, from Laplace's formulae for the functions P_n and Q_n,

$$P_n(x)\,Q_{n+1}(t)-P_{n+1}(x)\,Q_n(t)=\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\left\{\frac{x+(x^2-1)^{\frac{1}{2}}\cos\phi}{t+(t^2-1)^{\frac{1}{2}}\cosh\psi}\right\}^n A\,d\phi\,d\psi,$$

where A denotes a quantity which is finite and independent of n.

It is clear that this double-integral tends to zero only when, for all values of ϕ between zero and π, and all values of ψ between zero and infinity, the inequality

$$\left|\frac{x+(x^2-1)^{\frac{1}{2}}\cos\phi}{t+(t^2-1)^{\frac{1}{2}}\cosh\psi}\right|<1$$

is satisfied.

Writing $$x=\frac{1}{2}\left(u+\frac{1}{u}\right),\qquad t=\frac{1}{2}\left(v+\frac{1}{v}\right),$$

the inequality becomes

$$\left|u+\frac{1}{u}+\left(u-\frac{1}{u}\right)\cos\phi\right|<\left|v+\frac{1}{v}+\left(v-\frac{1}{v}\right)\cosh\psi\right|.$$

The left-hand side of this relation has its maximum value when $\cos\phi=1$, the value being $2\,|u|$.

The right-hand side similarly has a minimum value equal to $2\,|v|$.

The condition thus becomes

$$|u|<|v|$$

or $$|x+(x^2-1)^{\frac{1}{2}}|<|t+(t^2-1)^{\frac{1}{2}}|.$$

This inequality shews that *the point x must be in the interior of an ellipse, which passes through the point t, and which has the points $+1$, -1 for its foci*: for if a be the major axis of this ellipse, then

$$t=a\cos\theta+i\,(a^2-1)^{\frac{1}{2}}\sin\theta,$$

where θ is the eccentric angle of t in the ellipse; and thus

$$(t^2-1)^{\frac{1}{2}}=(a^2-1)^{\frac{1}{2}}\cos\theta+ia\sin\theta,$$

and $$t+(t^2-1)^{\frac{1}{2}}=\{a+(a^2-1)^{\frac{1}{2}}\}\,e^{i\theta},$$

so that $$|t+(t^2-1)^{\frac{1}{2}}|=a+(a^2-1)^{\frac{1}{2}},$$

and hence the above inequality shews that the semi-axis of the ellipse which passes through x is less than the semi-axis of the ellipse which passes through t, i.e. that x is within the ellipse which passes through t.

Hence *if the point x is in the interior of the ellipse which passes through the point t and has the points $+1$, -1, for its foci, then the expansion*

$$\frac{1}{t-x} = \sum_{n=0}^{\infty} (2n+1) P_n(x) Q_n(t)$$

is valid.

128. *Neumann's theorem on the expansion of an arbitrary function in a series of Legendre polynomials.*

We proceed now to discuss the expansion of any arbitrarily given function in terms of the polynomials of Legendre. The expansion is of special interest, inasmuch as it represents the case which stands next in simplicity to Taylor's series, among expansions in series of polynomials.

Let $f(z)$ be any function, which is regular at all points in the interior of an ellipse C, whose foci are at the points $z=-1$ and $z=+1$. We shall shew that it is possible to expand $f(z)$ in a series of the form

$$a_0 P_0(z) + a_1 P_1(z) + a_2 P_2(z) + a_3 P_3(z) + \dots,$$

where $a_0, a_1, a_2 \dots$ are independent of z: and that this expansion is valid for all points z in the interior of the ellipse C.

For let $z=t$ be any point on the circumference of the ellipse.

Then we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z} = \frac{1}{2\pi i}\int_C f(t)\,dt \sum_{n=0}^{\infty} (2n+1) P_n(z) Q_n(t),$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z),$$

where

$$a_n = \frac{2n+1}{2\pi i}\int_C f(t) Q_n(t)\,dt.$$

This is the required expansion.

Another form for a_n can be obtained in the following way.

We have

$$\begin{aligned} a_n &= \frac{2n+1}{2\pi i}\int_C f(t)\,dt \,.\, \frac{1}{2}\int_{-1}^{+1} P_n(y)\frac{dy}{t-y} \\ &= \frac{2n+1}{2}\int_{-1}^{+1} P_n(y)\,dy\, \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-y} \\ &= \frac{2n+1}{2}\int_{-1}^{+1} f(y) P_n(y)\,dy. \end{aligned}$$

The latter is the more usual form for a_n.

Example 1. Shew that the semi-axes of the ellipse, within which the series

$$\Sigma c_n P_n(z)$$

converges, are

$$\frac{1}{2}\left(\rho+\frac{1}{\rho}\right) \text{ and } \frac{1}{2}\left(\rho-\frac{1}{\rho}\right),$$

where ρ is the radius of convergence of the series

$$\Sigma c_n z^n.$$

Example 2. If

$$z=\left(\frac{y-1}{y+1}\right)^{\frac{1}{2}}, \text{ and } k^2=\frac{(x-1)(y+1)}{(x+1)(y-1)},$$

prove that

$$\int_z^1 \frac{dz}{\{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}}}=\{(x+1)(y-1)\}^{\frac{1}{2}} \sum_{n=0}^{\infty} P_n(x)\, Q_n(y).$$

129. *The associated functions $P_n^m(z)$ and $Q_n^m(z)$.*

We shall now introduce a more extended class of Legendre functions.

If m be any positive integer, the quantities

$$(1-z^2)^{\frac{m}{2}} \frac{d^m P_n(z)}{dz^m} \text{ and } (1-z^2)^{\frac{m}{2}} \frac{d^m Q_n(z)}{dz^m}$$

will be called *the associated Legendre functions of the nth degree and mth order*, and will be denoted by $P_n^m(z)$ and $Q_n^m(z)$ respectively.

We shall first shew that *the associated Legendre functions satisfy a differential equation analogous to the Legendre differential equation.*

For let the Legendre differential equation

$$(1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+n(n+1)y=0$$

be differentiated m times, and let v be written for $\dfrac{d^my}{dz^m}$.

We thus have for v the equation

$$(1-z^2)\frac{d^2v}{dz^2}-2z(m+1)\frac{dv}{dz}+(n-m)(n+m+1)v=0.$$

Write $$w=v(1-z^2)^{\frac{m}{2}};$$

the equation becomes

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{n(n+1)-\frac{m^2}{1-z^2}\right\}w=0.$$

This is the differential equation satisfied by the functions

$$P_n^m(z) \text{ and } Q_n^m(z).$$

Several expressions for the associated Legendre functions can be obtained easily from the above definitions.

Thus from Schläfli's formula, we have

$$P_n^m(z) = \frac{(n+1)(n+2)\dots(n+m)}{2\pi i} \frac{(1-z^2)^{\frac{m}{2}}}{2^n} \int_C (t^2-1)^n (t-z)^{-n-m-1}\, dt,$$

where C is a simple contour enclosing the points $t=1$ and $t=z$, but not enclosing the point $t=-1$.

From this result, or from Rodrigues' formula, we have, when n is a positive integer,

$$P_n^m(z) = (1-z^2)^{\frac{m}{2}} \frac{1}{2^n\, n!} \frac{d^{n+m}(z^2-1)^n}{dz^{n+m}}.$$

130. *The definite integrals of the associated Legendre functions.*

The theorems already given in § 115, relating to the definite integrals of the Legendre functions, can be generalised so as to be stated in the following form: *When m and n are positive integers,*

$$\int_{-1}^{+1} P_n^m(z)\, P_r^m(z)\, dz = 0, \qquad \text{when } r \lessgtr n,$$

and

$$\int_{-1}^{+1} \{P_n^m(z)\}^2\, dz = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$

To establish these results, we use the identity

$$\frac{d^{n-m}}{dz^{n-m}}(z^2-1)^n = \frac{(n-m)!}{(n+m)!}(z^2-1)^m \frac{d^{n+m}}{dz^{n+m}}(z^2-1)^n,$$

which gives

$$\int_{-1}^{1} \{P_n^m(z)\}^2\, dz = \int_{-1}^{1} \frac{1}{2^{2n}(n!)^2}(1-z^2)^m \left\{\frac{d^{n+m}(z^2-1)^n}{dz^{n+m}}\right\}^2 dz$$

$$= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \frac{(n+m)!}{(n-m)!}(-1)^m$$

$$\times \left\{\frac{d^{n+m}}{dz^{n+m}}(z^2-1)^n\right\} \left\{\frac{d^{n-m}}{dz^{n-m}}(z^2-1)^n\right\} dz$$

(integrating by parts) $$= \frac{1}{2^{2n}(n!)^2} \frac{(n+m)!}{(n-m)!} \int_{-1}^{1} \left\{\frac{d^n}{dz^n}(z^2-1)^n\right\}^2 dz$$

$$= \frac{(n+m)!}{(n-m)!} \int_{-1}^{1} \{P_n(z)\}^2\, dz$$

$$= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$

We can prove in the same way the other result stated, namely that

$$\int_{-1}^{+1} P_n^m(z)\, P_r^m(z)\, dz = 0, \qquad \text{when } r \neq n.$$

For this integral in the same manner reduces to a multiple of

$$\int_{-1}^{+1} P_n(z)\, P_r(z)\, dz,$$

which is zero when n and r are different.

131. *Expression of $P_n^m(z)$ as a definite integral of Laplace's type.*

The associated Legendre functions can be expressed by means of definite integrals of the same type as those found in § 119 and § 125, as will appear from the following investigation.

We have

$$\begin{aligned}
&\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\, d\phi \\
&\qquad = -\left[\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m-1}\phi\cos\phi\right] \\
&\qquad\quad + \int_0^\pi \cos\phi\,\frac{d}{d\phi}\left[\sin^{2m-1}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\right] d\phi \\
&\qquad = (2m-1)\int_0^\pi \cos^2\phi\sin^{2m-2}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\, d\phi \\
&\qquad\quad -(n-m)\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m-1}\cos\phi\,(z^2-1)^{\frac{1}{2}}\sin^{2m}\phi\, d\phi \\
&\qquad = (2m-1)\int_0^\pi \sin^{2m-2}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\, d\phi \\
&\qquad\quad -(2m-1)\int_0^\pi \sin^{2m}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\, d\phi \\
&\qquad\quad -(n-m)\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\, d\phi \\
&\qquad\quad +(n-m)\,z\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m-1}\sin^{2m}\phi\, d\phi.
\end{aligned}$$

We thus have

$$\begin{aligned}
&(n+m)\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\, d\phi \\
&\qquad = (2m-1)\int_0^\pi \sin^{2m-2}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\, d\phi \\
&\qquad\quad - \frac{z}{(z^2-1)^{\frac{1}{2}}}\left[\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m-1}\phi\right] \\
&\qquad\quad + \frac{z}{(z^2-1)^{\frac{1}{2}}}\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}(2m-1)\sin^{2m-2}\phi\cos\phi\, d\phi,
\end{aligned}$$

or $$\frac{n+m}{2m-1}\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\,d\phi$$

$$=\int_0^\pi \sin^{2m-2}\phi\,\{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\{1+z\,(z^2-1)^{-\frac{1}{2}}\cos\phi\}\,d\phi$$

$$=\frac{1}{n-m+1}\frac{d}{dz}\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m+1}\sin^{2m-2}\phi\,d\phi.$$

Thus if we write

$$I_m=\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\,d\phi,$$

we have $$I_m=\frac{(2m-1)}{(n+m)(n-m+1)}\frac{dI_{m-1}}{dz},$$

and therefore $$I_m=\frac{(2m-1)(2m-3)\dots 1}{(n+m)(n+m-1)\dots(n-m+1)}\frac{d^mI_0}{dz^m}.$$

But $$I_0=\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^n\,d\phi=\pi P_n(z),$$

when the real part of z is positive.

Therefore $$I_m=\frac{(2m-1)(2m-3)\dots 1\,.\,\pi}{(n+m)(n+m-1)\dots(n-m+1)}\frac{d^m}{dz^m}P_n(z)$$

$$=\frac{(2m-1)(2m-3)\dots 1\,.\,\pi}{(n+m)(n+m-1)\dots(n-m+1)}(1-z^2)^{-\frac{m}{2}}P_n^m(z),$$

or $$P_n^m(z)=\frac{(n+m)(n+m-1)\dots(n-m+1)}{(2m-1)(2m-3)\dots 1\,.\,\pi}(1-z^2)^{\frac{m}{2}}$$

$$\times\int_0^\pi \{z+\cos\phi\,(z^2-1)^{\frac{1}{2}}\}^{n-m}\sin^{2m}\phi\,d\phi.$$

This result expresses $P_n^m(z)$ as a definite integral of Laplace's type, valid for all values of n when the real part of z is positive.

132. *Alternative expression of $P_n^m(z)$ as a definite integral of Laplace's type.*

The formula last found can be replaced by another result, found in the following way.

If in Jacobi's well-known theorem*

$$\int_0^\pi f(\cos\phi)\cos m\phi\,d\phi=\frac{1}{1\,.\,3\,.\,5\dots(2m-1)}\int_0^\pi f^{(m)}(\cos\phi)\sin^{2m}\phi\,d\phi,$$

we take $$f(\cos\phi)=\{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^n,$$

* *Crelle's Journal*, XV.

so that

$$f^{(m)}(\cos\phi) = n(n-1)\dots(n-m+1)(z^2-1)^{\frac{m}{2}}\{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^{n-m},$$

we obtain

$$\int_0^\pi \{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^n \cos m\phi\, d\phi$$

$$= \frac{n(n-1)\dots(n-m+1)}{1.3.5\dots(2m-1)}(z^2-1)^{\frac{m}{2}}$$

$$\times \int_0^\pi \{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^{n-m}\sin^{2m}\phi\, d\phi$$

$$= \frac{\pi(-1)^{\frac{m}{2}}}{(n+m)(n+m-1)\dots(n+1)} P_n^m(z).$$

Therefore

$$P_n^m(z) = \frac{(n+m)(n+m-1)\dots(n+1)}{\pi}(-1)^{\frac{m}{2}}$$

$$\times \int_0^\pi \{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^n \cos m\phi\, d\phi.$$

This formula is valid for all values of n, and for all values of z whose real part is positive; m being a positive integer.

133. *The function $C_n^\nu(z)$.*

A function connected with the associated Legendre functions $P_n^m(z)$ is the function $C_n^\nu(z)$, which for integral values of ν is defined to be the coefficient of h^n in the expansion, in ascending powers of h, of the quantity

$$(1-2hz+h^2)^{-\nu}.$$

It is easily seen that $C_n^\nu(z)$ satisfies the differential equation

$$\frac{d^2y}{dz^2} + \frac{(2\nu+1)z}{z^2-1}\frac{dy}{dz} - \frac{n(n+2\nu)}{z^2-1}y = 0.$$

For all values of n and ν, it may be shewn that $C_n^\nu(z)$ can be defined by a contour-integral of the form

$$\text{Constant} \times (1-z^2)^{\frac{1}{2}-\nu}\int_C \frac{(1-t^2)^{n+\nu-\frac{1}{2}}}{(t-z)^{n+1}}dt.$$

When n is integral, we have

$$C_n^\nu(z) = \frac{(-2)^n \nu(\nu+1)\dots(\nu+n-1)}{n!\,(2n+2\nu-1)(2n+2\nu-2)\dots(n+2\nu)}$$

$$\times (1-z^2)^{\frac{1}{2}-\nu}\frac{d^n}{dz^n}\{(1-z^2)^{n+\nu-\frac{1}{2}}\},$$

which corresponds to Rodrigues' formula for $P_n(z)$; in fact, since

$$P_n(z) = C_n^{\frac{1}{2}}(z),$$

Rodrigues' formula is a particular case of this formula.

When r is an integer, we have

$$C_{n-r}^{r+\frac{1}{2}}(z)=\frac{1}{(2r-1)(2r-3)\dots 3.1}\frac{d^r}{dz^r}P_n(z),$$

whence we have

$$C_{n-r}^{r+\frac{1}{2}}(z)=\frac{(1-z^2)^{-\frac{r}{2}}}{(2r-1)(2r-3)\dots 3.1}P_n^r(z).$$

The last equation gives the connexion between the functions $C_n^\nu(z)$ and $P_n^r(z)$.

This function $C_n^\nu(z)$ has the following further properties, analogous to the recurrence-formulae,

$$zC_{n-1}^{\nu+1}(z)-C_{n-2}^{\nu+1}(z)-\frac{n}{2\nu}C_n^\nu(z)=0,$$

$$\frac{dC_n^\nu(z)}{dz}=2\nu C_{n-1}^{\nu+1}(z),$$

$$C_n^{\nu+1}(z)-zC_{n-1}^{\nu+1}(z)=\frac{n+2\nu}{2\nu}C_n^\nu(z),$$

$$nC_n^\nu(z)=(n-1+2\nu)\,z\,C_{n-1}^\nu(z)-2\nu(1-z^2)\,C_{n-2}^{\nu-1}(z).$$

Miscellaneous Examples.

1. Shew that when n is a positive integer,

$$P_n(z)=\frac{(-1)^n}{n!}\frac{d^n}{dz^n}(u^2+z^2)^{-\frac{1}{2}},$$

where u^2 is to be replaced by $(1-z^2)$ after the differentiation has been performed.

2. Prove that when n is a positive integer,

$$P_n(z)=\sum_0^n\frac{(n+p)!\;(-1)^p}{(n-p)!\;p!\;p!\;2^{p+1}}\{(1-z)^p+(-1)^n(1+z)^p\}.$$

(Cambridge Mathematical Tripos, Part I, 1898.)

3. Shew that

$$\sin^{-1}z=\frac{\pi}{2}\sum_0^\infty\left\{\frac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}\right\}^2\{P_{2n+1}(z)-P_{2n-1}(z)\}.$$

(Catalan.)

4. Prove that

$$\int_{-1}^1 z(1-z^2)\frac{dP_n}{dz}\frac{dP_m}{dz}dz$$

is zero unless $m-n=\pm 1$, and determine its value in these cases.

(Cambridge Mathematical Tripos, Part I, 1896.)

5. Shew (by induction or otherwise) that when n is a positive integer,

$$(2n+1)\int_z^1 P_n^2(z)\,dz=1-zP_n^2-2z(P_1^2+P_2^2+\dots+P_{n-1}^2)+2(P_1P_2+P_2P_3+\dots+P_{n-1}P_n).$$

(Cambridge Mathematical Tripos, Part I, 1899.)

6. Shew that, if k is an odd number,

$$\frac{1}{(1-2zh+h^2)^{\frac{k}{2}}}=\Sigma a_nP_n(z),$$

where

$$a_n = \frac{h^n}{(1-h^2)^{k-2}} \frac{2^{\frac{1}{2}(k-3)}(2n+1)}{1.3.5\ldots(k-2)} \left(h^2\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{\frac{1}{2}(k-3)} \times x^{-\frac{1}{2}(2n-k+4)}\, y^{\frac{1}{2}(2n+k-2)},$$

where x and y are to be replaced by unity after the differentiations have been performed.

(Routh.)

7. If

$$(h^3-3hz+1)^{-\frac{1}{3}} = \sum_{n=0}^{\infty} R_n(z)\, h^n,$$

shew that

$$2(n+1)R_{n+1} - 3(2n+1)R_n + (2n-1)R_{n-2} = 0$$

and

$$nR_n + R'_{n-2} - zR_n' = 0$$

and

$$4(4z^3-1)R_n''' + 96z^2R_n'' - z(12n^2+24n-91)R_n' - n(2n+3)(2n+9)R_n = 0,$$

where

$$R_n''' = \frac{d^3R_n}{dz^3}, \text{ etc.}$$

(Pincherle.)

8. If m and n be positive integers, and $m \leqslant n$, shew that

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \frac{A_{m-r}A_rA_{n-r}}{A_{n+m-r}}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}(z),$$

where

$$A_m = \frac{1.3.5\ldots(2m-1)}{m!}.$$

(Adams.)

9. Shew that $P_n(z)$ can be expressed as a determinant in which all elements parallel to the auxiliary diagonal are equal (i.e. all elements are equal for which the sum of the row-index and column-index is the same); the determinant containing $(2n-1)$ rows, and its first row being

$$z,\ -\frac{1}{3},\ \frac{1}{3}z,\ -\frac{1}{5},\ \frac{1}{5}z, \ldots \frac{1}{2n-1}z.$$

(Heun.)

10. Shew that

$$P_n(z) = \frac{2}{\pi i}\int_0^{\infty} \frac{\{z(1-t^2)-2t(1-z^2)^{\frac{1}{2}}\}^n}{(1-t^2)^{n+1}}\,dt.$$

(Silva.)

11. Shew that

$$\int_{-1}^{1} \frac{1}{z-x}\{P_n(x)P_{n-1}(z) - P_{n-1}(x)P_n(z)\}\,dx = -\frac{z}{n},$$

$$\sum_{1}^{\infty} \frac{1}{2n+1}\frac{d}{dz}\left[P_n\left(\frac{1}{n}P_{n-1} + \frac{1}{n+1}P_{n+1}\right)\right] = -1.$$

(Catalan.)

12. Shew that, when n is a positive integer,

$$\frac{Q_n(\cos\theta)}{r^{2n+1}} = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{2r}\log\left(\frac{r-z}{r+z}\right)\right\},$$

where $z = r\cos\theta$.

13. Shew that the complete solution of the Legendre differential equation is

$$y = AP_n(z) + BP_n(z)\int_0^z \frac{d\mu}{(1-\mu^2)\{P_n(\mu)\}^2}.$$

14. Shew that

$$\{z+(z^2-1)^{\frac{1}{2}}\}^a = \sum_{m=0}^{\infty} B_m Q_{2m-a-1}(z),$$

where

$$B_m = -\frac{a(a+2m+\frac{1}{2})}{2\pi}\,\frac{\Gamma(m-\frac{1}{2})\,\Gamma(m-a-\frac{1}{2})}{m!\;\Gamma(m-a+1)}.$$

15. Shew that, when the real part of $(n+1)$ is positive,

$$Q_n(z) = \int_{z+(z^2-1)^{\frac{1}{2}}}^{\infty} \frac{h^{-n-1}}{(1-2hz+h^2)^{\frac{1}{2}}}\,dh,$$

and

$$Q_n(z) = \int_0^{z-(z^2-1)^{\frac{1}{2}}} \frac{h^n}{(1-2hz+h^2)^{\frac{1}{2}}}\,dh.$$

16. Prove that

$$P_{n+1}(z)\,Q_{n-1}(z) - P_{n-1}(z)\,Q_{n+1}(z) = \frac{2n+1}{n(n+1)}\,z.$$

(Cambridge Mathematical Tripos, Part II, 1894.)

17. Shew that, if n be a positive integer,

$$Q_n(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).\,P_n(z) - \frac{1}{2}\left\{P_{n-1}(z)\,P_0(z) + \frac{1}{2}P_{n-2}(z)\,P_1(z) + \frac{1}{3}P_{n-3}(z)\,P_2(z) + \dots\right\}.$$

18. Shew that

$$Q_n(z) = \frac{1}{2}P_n(z)\log\frac{z+1}{z-1} - \left\{\frac{2n-1}{1\,.\,n}P_{n-1}(z) + \frac{2n-5}{3(n-1)}P_{n-3}(z) + \frac{2n-9}{5(n-2)}P_{n-5}(z) + \dots\right\},$$

and

$$Q_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left\{(z^2-1)^n\log\frac{z+1}{z-1}\right\} - \frac{1}{2}P_n(z)\log\frac{z+1}{z-1},$$

where n is a positive integer, and $z > 1$, and where $\log\frac{z+1}{z-1}$ is to be changed into $\log\frac{1+z}{1-z}$ if z is numerically less than unity.

Prove also that

$$Q_n(z) = \frac{1}{2}P_n(z)\log\frac{z+1}{z-1}$$

$$-\left\{\begin{array}{l} k + (k-1)\dfrac{n(n+1)}{1^2}\left(\dfrac{z-1}{2}\right) + \left(k-1-\dfrac{1}{2}\right)\dfrac{n(n-1)(n+1)(n+2)}{1^2\,2^2}\left(\dfrac{z-1}{2}\right)^2 \\ \quad + \left(k-1-\dfrac{1}{2}-\dfrac{1}{3}\right)\dfrac{n(n-1)(n-2)(n+1)(n+2)(n+3)}{1^2\,2^2\,3^2}\left(\dfrac{z-1}{2}\right)^3 + \dots \end{array}\right\},$$

where $k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

(Cambridge Mathematical Tripos, Part II, 1898.)

19. Shew that

$$\frac{1}{2} P_n^m(z) = \frac{n(n+1)\dots(n+m-1)}{m!} z^m F(n,\ n+m,\ m+1,\ z^2).$$

20. Prove that, if

$$y_s = \frac{(2n+1)(2n+3)\dots(2n+2s-1)}{n(n^2-1)(n^2-4)\dots\{n^2-(s-1)^2\}(n+s)} (z^2-1)^s \frac{d^s P_n}{dz^s},$$

then

$$y_2 = P_{n+2} - \frac{2(2n+1)}{2n-1} P_n + \frac{2n+3}{2n-1} P_{n-2},$$

$$y_3 = P_{n+3} - \frac{3(2n+3)}{2n-1} P_{n+1} + \frac{3(2n+5)}{2n-3} P_{n-1} - \frac{(2n+3)(2n+5)}{(2n-1)(2n-3)} P_{n-3},$$

and find the general formula.

(Cambridge Mathematical Tripos, Part II, 1896.)

21. If

$$\frac{1}{(1-2hz+h^2)^\nu} = \sum_{n=0}^{\infty} C_n^\nu(z) h^n,$$

shew that

$$C_n^\nu\{xx_1 - (x^2-1)^{\frac{1}{2}}(x_1^2-1)^{\frac{1}{2}}\cos\phi\}$$

$$= \frac{\Pi(2\nu-2)}{\{\Pi(\nu-1)\}^2} \sum_{\lambda=0}^{\lambda=n} (-1)^\lambda \frac{4^\lambda \Pi(n-\lambda)\{\Pi(\nu+\lambda-1)\}^2}{\Pi(n+2\nu+\lambda-1)}$$

$$\times (x^2-1)^{\frac{1}{2}\lambda}(x_1^2-1)^{\frac{1}{2}\lambda} C_{n-\lambda}^{\nu+\lambda}(x)\, C_{n-\lambda}^{\nu+\lambda}(x_1)\, C_\lambda^{\frac{1}{2}(2\nu-1)}(\cos\phi).$$

(Gegenbauer.)

22. If

$$\sigma_n(z) = \int_0^{e_1} (t^3 - 3tz + 1)^{-\frac{1}{2}} t^n dt,$$

where e_1 is the least root of $t^3 - 3tz + 1 = 0$, shew that

$$(2n+1)\sigma_{n+1} - 3(2n-1) z\sigma_{n-1} + 2(n-1)\sigma_{n-2} = 0,$$

and

$$4(4z^3-1)\sigma_n''' + 144z_2\sigma_n'' - z(12n^2 - 24n - 291)\sigma_n' - (n-3)(2n-7)(2n+5)\sigma_n = 0,$$

where

$$\sigma_n''' = \frac{d^3\sigma_n(z)}{dz^3}, \text{ etc.}$$

(Pincherle.)

23. Shew that

$$Q_n^m(z) = e^{m\pi i} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \int_0^\infty \frac{\cosh mu}{\{z+(z^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}} du,$$

where the real part of $(n+1)$ is greater than m.

(Hobson.)

24. The equation of a nearly spherical surface of revolution is

$$r = 1 + a\{P_1(\cos\theta) + P_3(\cos\theta) + \dots + P_{2n-1}(\cos\theta)\},$$

where a is small; shew that to the first order of a the radius of curvature of the meridian is

$$1 + a\sum_{m=0}^{n-1} \{n(4m+3) - (m+1)(8m+3)\} P_{2m+1}(\cos\theta).$$

(Cambridge Mathematical Tripos, Part I, 1894.)

CHAPTER XI.

HYPERGEOMETRIC FUNCTIONS.

134. *The Hypergeometric Series.*

We have already in § 14 considered the *hypergeometric series**

$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \dots$$

from the point of view of its convergence. It was there shewn that the series is absolutely convergent for all values of z represented by points in the interior of the circle whose centre is at the origin and whose radius is unity. It follows from § 22 that all the series which can be derived from the hypergeometric series by differentiation and integration are likewise absolutely convergent within the same region: and by § 55, the convergence is not only absolute but uniform over the interior of the circle, and the sums of the series obtained by differentiation and integration of the series term by term are the derivates and integrals respectively of the sum of the series. The hypergeometric series, together with the series which can be derived from it by the process of continuation (§ 41), will therefore represent an analytic function of the variable z; this function will be denoted by $F(a, b, c, z)$.

Many of the most important functions of Analysis can be expressed by means of the hypergeometric series. Thus it is easily seen that

$$(1+z)^n = F(-n, \beta, \beta, -z),$$

$$\log(1+z) = zF(1, 1, 2, -z),$$

$$e^z = \underset{\beta=\infty}{\text{Limit}}\, F\left(1, \beta, 1, \frac{z}{\beta}\right),$$

and we have shewn in the preceding chapter that the Legendre functions may be represented by the series

* The name was given by Wallis in 1655.

$$P_n(z) = F\left(-n,\ n+1,\ 1,\ \frac{1-z}{2}\right),$$

$$P_n(z) = \frac{2^n z^n \Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+1)} F\left(\frac{1-n}{2},\ -\frac{n}{2},\ \frac{1}{2}-n,\ \frac{1}{z^2}\right),$$

$$Q_n(z) = \frac{\pi^{\frac{1}{2}}\Gamma(n+1)}{2^{n+1}\Gamma(n+\frac{3}{2})}\frac{1}{z^{n+1}} F\left(\frac{n+1}{2},\ \frac{n+2}{2},\ n+\frac{3}{2},\ \frac{1}{z^2}\right).$$

These examples are sufficient to shew that the functions represented by the hypergeometric series are in some cases one-valued and in other cases many-valued.

Example. Shew that

$$\frac{d}{dz}F(a,\ b,\ c,\ z) = \frac{ab}{c}F(a+1,\ b+1,\ c+1,\ z).$$

135. *Value of the series $F(a, b, c, 1)$.*

We have shewn in § 14 that the series $F(a, b, c, 1)$ converges absolutely so long as the real part of $c-a-b$ is positive. Suppose this condition to be satisfied. Then we have

$$F(a, b, c, 1) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{n!\,\Gamma(c+n)\Gamma(a)\Gamma(b)}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(n+a)\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma(n+a)\, B(n+b, c-b)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty e^{-x} x^{a-1+n}\,dx \int_0^1 (1-z)^{c-b-1} z^{b-1+n}\,dz$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_0^\infty e^{-x} x^{a-1}\,dx \int_0^1 e^{xz}(1-z)^{c-b-1} z^{b-1}\,dz.$$

Writing $z = 1-t$, this becomes

$$F(a, b, c, 1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_0^\infty dx \int_0^1 e^{-tx} x^{a-1} t^{c-b-1} (1-t)^{b-1}\,dt$$

(writing $xt = s$)

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_0^\infty ds \int_0^1 e^{-s} s^{a-1} t^{c-a-b-1} (1-t)^{b-1}\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \Gamma(a)\, B(c-a-b, b)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

The hypergeometric series with argument unity can thus be expressed in terms of Gamma-functions.

136. *The differential equation satisfied by the hypergeometric series.*

The function represented by the hypergeometric series $y = F(a, b, c, z)$ satisfies the differential equation

$$z(z-1)\frac{d^2y}{dz^2} + \{-c + (a+b+1)z\}\frac{dy}{dz} + aby = 0;$$

for if the series be substituted for y in the left-hand side of the equation, the coefficient of z^r is

$$\frac{a(a+1)\ldots(a+r-1)\,b(b+1)\ldots(b+r-1)}{1.2\ldots r.c(c+1)\ldots(c+r)}$$

$$\{r(r-1)(c+r) - r(a+r)(b+r) - c(a+r)(b+r) + r(c+r)(a+b+1) + ab(c+r)\}$$

or zero; which establishes the result.

Example. Shew that one integral of the equation

$$z^2(z-1)\frac{d^2y}{dz^2} + z(az+b)\frac{dy}{dz} + (cz+f)y = 0$$

is

$$z^m F(m-\mu,\ m-\nu,\ m-n+1,\ z),$$

where

$$a - 1 = -(\mu+\nu),$$

$$c = \mu\nu,$$

$$b + 1 = m + n,$$

$$f = -mn.$$

137. *The differential equation of the general hypergeometric function.*

The differential equation found in the preceding article is a case of a more general differential equation, which may be written

$$\frac{d^2y}{dz^2} + \left\{\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right\}\frac{dy}{dz}$$

$$+ \left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-a)(b-c)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}\frac{y}{(z-a)(z-b)(z-c)} = 0 \qquad \ldots(A),$$

in which $a, b, c, \alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are any constants such that the equation

$$\alpha + \beta + \gamma + \alpha' + \beta' + \gamma' = 1$$

is satisfied. This will be called *the differential equation of the general hypergeometric function.* The form here given is due to Papperitz*.

* *Math. Annalen*, xxv.

We shall now shew that the differential equation satisfied by the hypergeometric series is a particular case of this equation.

For in the equation (A), write

$$a=0, \quad b=\infty, \quad c=1.$$

The equation becomes

$$\frac{d^2y}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z}+\frac{1-\gamma-\gamma'}{z-1}\right\}\frac{dy}{dz}+\left\{-\frac{\alpha\alpha'}{z}+\frac{\gamma\gamma'}{z-1}+\beta\beta'\right\}\frac{y}{z(z-1)}=0.$$

In this equation, let α and γ be replaced by zero. We thus have

$$\frac{d^2y}{dz^2}+\left(\frac{1-\alpha'}{z}+\frac{1-\gamma'}{z-1}\right)\frac{dy}{dz}+\frac{\beta\beta' y}{z(z-1)}=0,$$

and in this equation the constants $\alpha', \gamma', \beta, \beta'$, are to be such as to satisfy the relation

$$\beta+\alpha'+\beta'+\gamma'=1.$$

This differential equation can be identified with the equation

$$z(z-1)\frac{d^2y}{dz^2}+\{-c+(a+b+1)z\}\frac{dy}{dz}+aby=0,$$

which is the differential equation satisfied by the hypergeometric series, by writing

$$\beta=a, \quad \beta'=b, \quad \alpha'=1-c;$$

which in virtue of the above relation gives $\gamma'=c-a-b$. The differential equation of the hypergeometric series is therefore a special case of equation (A).

We shall denote any solution of the general differential equation (A) by the symbol

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

This notation is due to Riemann*; it enables us to express our result thus:

The hypergeometric series

$$F(a, b, c, z)$$

is a solution of the differential equation of the class of functions

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}.$$

* *Abhandlungen d. K. Gesell. d. Wissenschaften zu Göttingen*, VII. (1857).

Although the hypergeometric series itself satisfies only a particular form of the differential equation (A), it is nevertheless possible to satisfy the general equation (A) by means of a function derived from the hypergeometric function. For by the transformation

$$x(z-b)(c-a) = (z-a)(c-b),$$

the differential equation (A) is reduced to the form

$$\frac{d^2y}{dx^2} + \left\{\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right\}\frac{dy}{dx} + \left\{-\frac{\alpha\alpha'}{x} + \frac{\gamma\gamma'}{x-1} + \beta\beta'\right\}\frac{y}{x(x-1)} = 0.$$

In this we take a new dependent variable, defined by the equation

$$y = x^\alpha(1-x)^\gamma u.$$

The equation becomes

$$\frac{d^2u}{dx^2} + \left(\frac{1-\alpha'+\alpha}{x} + \frac{1-\gamma'+\gamma}{x-1}\right)\frac{du}{dx} + \{\beta\beta' + (\alpha+\gamma)(1-\alpha'-\gamma')\}\frac{u}{x(x-1)} = 0.$$

Now the equation

$$\alpha + \beta + \gamma + \alpha' + \beta' + \gamma' = 1$$

will be satisfied if $\beta, \alpha', \beta', \gamma'$, are expressible in terms of three new constants, a, b, c, defined by the formulae

$$\begin{cases} \beta = a - \alpha - \gamma, \\ \alpha' = 1 - c + \alpha, \\ \beta' = b - \alpha - \gamma, \\ \gamma' = c - a - b + \gamma. \end{cases}$$

The differential equation for u can now be written

$$x(x-1)\frac{d^2u}{dx^2} + \{(1+a+b)x - c\}\frac{du}{dx} + abu = 0.$$

But this is the differential equation satisfied by the hypergeometric series, a solution of it being

$$F(a, b, c, x).$$

Hence we have, as one solution of the equation,

$$u = F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ x),$$

or $$y = x^\alpha(1-x)^\gamma F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ x),$$

or, disregarding a constant factor,

$$y = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma F\left(\alpha+\beta+\gamma, \alpha+\beta'+\gamma', 1+\alpha-\alpha', \frac{z-a\,.\,c-b}{z-b\,.\,c-a}\right).$$

This is therefore a solution, expressed by a hypergeometric series, of the differential equation which defines the class of functions

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

The advantage of the differential equation (A) over the equation found in § 136, which is satisfied by the hypergeometric series, lies in its greater symmetry and generality. The points $z=a$, $z=b$, and $z=c$, are called the *singularities* of the differential equation (A); the quantities α and α' are called the *exponents* at the singularity a; and similarly β and β' are the exponents at b, and γ and γ' are the exponents at c.

Example. Shew that

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \beta & \gamma & z^2 \\ \frac{1}{2} & \beta' & \gamma' & \end{matrix}\right\}=P\left\{\begin{matrix} -1 & \infty & 1 & \\ \gamma & 2\beta & \gamma & z \\ \gamma' & 2\beta' & \gamma' & \end{matrix}\right\}.$$

(Riemann.)

This relation follows from the fact that the differential equation corresponding to either of the P-functions is

$$\frac{d^2y}{dz^2}+\frac{2z(1-\gamma-\gamma')}{z^2-1}\frac{dy}{dz}+\left\{\beta\beta'+\frac{\gamma\gamma'}{z^2-1}\right\}\frac{4y}{z^2-1}=0.$$

138. *The Legendre functions as a particular case of the hypergeometric function.*

The expressions which have been found for $P_n(z)$ and $Q_n(z)$ as hypergeometric series naturally lead us to suppose that Legendre's differential equation is a special case of the differential equation which defines the general hypergeometric function. That this is the case appears from the following investigation.

If in equation (A) of the last article we take

$$a=-1,\quad b=\infty,\quad c=1,$$

we obtain the differential equation

$$\frac{d^2y}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z+1}+\frac{1-\gamma-\gamma'}{z-1}\right\}\frac{dy}{dz}+\left\{-\frac{2\alpha\alpha'}{z+1}+\beta\beta'+\frac{2\gamma\gamma'}{z-1}\right\}\frac{y}{(z-1)(z+1)}=0.$$

If now in this equation we take $\alpha=0$, $\alpha'=0$, $\gamma=0$, $\gamma'=0$, $\beta=n+1$, $\beta'=-n$, we obtain

$$\frac{d^2y}{dz^2}+\frac{2z}{z^2-1}\frac{dy}{dz}-n(n+1)\frac{y}{z^2-1}=0,$$

or

$$(1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+n(n+1)y=0,$$

which is the Legendre differential equation.

It follows from this that any solution of Legendre's equation is a hypergeometric function of the type

$$P\begin{Bmatrix} -1 & \infty & 1 & \\ 0 & n+1 & 0 & z \\ 0 & -n & 0 & \end{Bmatrix}.$$

In the same way it can be shewn that the associated Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ are hypergeometric functions of the type

$$P\begin{Bmatrix} -1 & \infty & 1 & \\ \frac{m}{2} & n+1 & \frac{m}{2} & z \\ -\frac{m}{2} & -n & -\frac{m}{2} & \end{Bmatrix}.$$

Example 1. Shew that

$$\frac{d^r}{dz^r} P_n(z) = P\begin{Bmatrix} -1 & \infty & 1 & \\ -r & n+r+1 & -r & z \\ 0 & -n+r & 0 & \end{Bmatrix}.$$

Example 2. If $z^2=\eta$, shew that the Legendre differential equation takes the form

$$\frac{d^2y}{d\eta^2}+\left\{\frac{1}{2\eta}-\frac{1}{1-\eta}\right\}\frac{dy}{d\eta}+\frac{n(n+1)y}{4\eta(1-\eta)}=0$$

Shew that this is a hypergeometric differential equation.

139. *Transformations of the general hypergeometric function.*

We shall next consider the effect of performing certain transformations in connexion with the general hypergeometric function

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

The differential equation satisfied by this function is

$$\frac{d^2y}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right\}\frac{dy}{dz}+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a}\right.$$

$$\left.+\frac{\beta\beta'(b-a)(b-c)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}\frac{y}{(z-a)(z-b)(z-c)}=0.$$

In this equation, let the dependent variable be changed by the transformation

$$y=\left(\frac{z-b}{z-a}\right)^{\delta} y'.$$

The differential equation for y' is found after a slight reduction to be

$$\frac{d^2y'}{dz^2}+\left\{\frac{1-\alpha-\alpha'-2\delta}{z-a}+\frac{1-\beta-\beta'+2\delta}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right\}\frac{dy'}{dz}$$

$$+\left\{\frac{(\alpha+\delta)(\alpha'+\delta)(a-b)(a-c)}{z-a}+\frac{(\beta-\delta)(\beta'-\delta)(b-c)(b-a)}{z-b}\right.$$

$$\left.+\gamma\gamma'\frac{(c-a)(c-b)}{z-c}\right\}\frac{y'}{(z-a)(z-b)(z-c)}=0.$$

This is the differential equation of a hypergeometric function which has exponents $\alpha+\delta$, $\alpha'+\delta$, at the singularity a, and exponents $\beta-\delta$, $\beta'-\delta'$, at the singularity b; and so we have

$$\left(\frac{z-a}{z-b}\right)^{\delta}P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & z \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix};$$

and hence in general we shall have

$$\left(\frac{z-a}{z-b}\right)^{\delta}\left(\frac{z-c}{z-b}\right)^{\epsilon}P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta-\epsilon & \gamma+\epsilon & z \\ \alpha'+\delta & \beta'-\delta-\epsilon & \gamma'+\epsilon & \end{Bmatrix}.$$

It will be observed that by this transformation the exponent-differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ are unaltered.

Consider now the effect of transformations of the independent variable z.

If we introduce in place of z a new variable z', defined by the equation

$$z=\frac{a_1z'+b_1}{c_1z'+d_1},$$

where a_1, b_1, c_1, d_1 are constants, so that

$$z'=\frac{-d_1z+b_1}{c_1z-a_1},$$

we have

$$\frac{dy}{dz}=\frac{a_1d_1-b_1c_1}{(c_1z-a_1)^2}\frac{dy}{dz'}=\frac{(c_1z'+d_1)^2}{a_1d_1-b_1c_1}\frac{dy}{dz'}$$

and

$$\frac{d^2y}{dz^2}=-\frac{2c_1(a_1d_1-b_1c_1)}{(c_1z-a_1)^3}\frac{dy}{dz'}+\frac{(a_1d_1-b_1c_1)^2}{(c_1z-a_1)^4}\frac{d^2y}{dz'^2}$$

$$=\frac{2c_1(c_1z'+d_1)^3}{(a_1d_1-b_1c_1)^2}\frac{dy}{dz'}+\frac{(c_1z'+d_1)^4}{(a_1d_1-b_1c_1)^2}\frac{d^2y}{dz'^2}.$$

Hence if we define quantities a', b', c' by the relations

$$a = \frac{a_1 a' + b_1}{c_1 a' + d_1}, \quad b = \frac{a_1 b' + b_1}{c_1 b' + d_1}, \quad c = \frac{a_1 c' + b_1}{c_1 c' + d_1},$$

so that

$$z - a = \frac{(a_1 d_1 - b_1 c_1)(z' - a')}{(c_1 z' + d_1)(c_1 a' + d_1)},$$

the general hypergeometric differential equation becomes

$$\frac{d^2 y}{dz'^2} + \frac{1}{c_1 z' + d_1}\frac{dy}{dz'}\left\{2c_1 + \frac{(1-\alpha-\alpha')(c_1 a' + d_1)}{z' - a'} + \frac{(1-\beta-\beta')(c_1 b' + d_1)}{z' - b'}\right.$$
$$\left. + \frac{(1-\gamma-\gamma')(c_1 c' + d_1)}{z' - c'}\right\} + \left\{\frac{\alpha\alpha'(a'-b')(a'-c')}{z'-a'} + \frac{\beta\beta'(b'-a')(b'-c')}{z'-b'}\right.$$
$$\left. + \frac{\gamma\gamma'(c'-a')(c'-b')}{z'-c'}\right\}\frac{y}{(z'-a')(z'-b')(z'-c')} = 0.$$

The coefficient of $\dfrac{dy}{dz'}$ in this equation can be written in the form

$$\frac{1-\alpha-\alpha'}{z'-a'} + \frac{1-\beta-\beta'}{z'-b'} + \frac{1-\gamma-\gamma'}{z'-c'} + \frac{1}{c_1 z' + d_1}\left\{\begin{matrix} 2c_1 - (1-\alpha-\alpha')c_1 \\ -(1-\beta-\beta')c_1 - (1-\gamma-\gamma')c_1 \end{matrix}\right\},$$

which, in virtue of the relation

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1,$$

reduces to

$$\frac{1-\alpha-\alpha'}{z'-a'} + \frac{1-\beta-\beta'}{z'-b'} + \frac{1-\gamma-\gamma'}{z'-c'}.$$

Hence the differential equation reduces to the differential equation of the function

$$P\left\{\begin{matrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\};$$

and thus we have the relation

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

This shews that *the general hypergeometric function is unaltered if the quantities a, b, c, z are replaced by quantities a', b', c', z', which are derived from them by the same homographic transformation.*

140. *The twenty-four particular solutions of the hypergeometric differential equation.*

We have seen in § 137 that a particular solution of the general hypergeometric differential equation is

$$\left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma',\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}.$$

We shall suppose that no one of the exponent-differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ is zero: it is shewn in treatises on Linear Differential Equations that when this exceptional case occurs, the general solution of the differential equation involves logarithmic terms; the formulae will be found in a memoir* by Lindelöf, to which the reader is referred.

Now if α be interchanged with α', or γ with γ', in this expression, it must still satisfy the differential equation, since the latter would be unaffected by this change. We thus obtain altogether four expressions for which

$$\frac{(c-b)(z-a)}{(c-a)(z-b)}$$

is the argument of the hypergeometric series, namely

$$y_1=\left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma',\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\},$$

$$y_2=\left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha'+\beta+\gamma,\ \alpha'+\beta'+\gamma',\ 1+\alpha'-\alpha,\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\},$$

$$y_3=\left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\},$$

$$y_4=\left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha'+\beta+\gamma',\ \alpha'+\beta'+\gamma,\ 1+\alpha'-\alpha,\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\};$$

these are all solutions of the differential equation.

Moreover, the differential equation is unaltered if the quantities α, α', a are interchanged respectively with β, β', b, or with γ, γ', c. If therefore we make such changes in the above solutions, they will still be solutions of the differential equation.

Let a change in which $(\alpha,\ \alpha',\ a)$ are interchanged with $(\beta,\ \beta',\ b)$ be denoted for example by

$$\begin{pmatrix} a, & b, & c \\ b, & a, & c \end{pmatrix},$$

each singularity in the bracket being interchanged with the singularity above or below it. Then there are five such changes possible, namely,

$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix},\ \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix},\ \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix},\ \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix},\ \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}.$$

* *Acta Soc. Scient. Fennicae*, XIX. (1893).

To each such change correspond (by interchanging α with α', etc. as already explained) four new solutions of the differential equation. We thus obtain twenty new solutions, which with the original four make altogether twenty-four particular solutions of the hypergeometric differential equation, in the form of hypergeometric series.

The twenty new solutions may be written down as follows:

$$\begin{cases} y_5 = \left(\dfrac{z-b}{z-c}\right)^{\beta}\left(\dfrac{z-a}{z-c}\right)^{\alpha} F\left\{\beta+\gamma+\alpha,\ \beta+\gamma'+\alpha',\ 1+\beta-\beta',\ \dfrac{(a-c)(z-b)}{(a-b)(z-c)}\right\}, \\ y_6 = \left(\dfrac{z-b}{z-c}\right)^{\beta'}\left(\dfrac{z-a}{z-c}\right)^{\alpha} F\left\{\beta'+\gamma+\alpha,\ \beta'+\gamma'+\alpha',\ 1+\beta'-\beta,\ \dfrac{(a-c)(z-b)}{(a-b)(z-c)}\right\}, \\ y_7 = \left(\dfrac{z-b}{z-c}\right)^{\beta}\left(\dfrac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta+\gamma+\alpha',\ \beta+\gamma'+\alpha,\ 1+\beta-\beta',\ \dfrac{(a-c)(z-b)}{(a-b)(z-c)}\right\}, \\ y_8 = \left(\dfrac{z-b}{z-c}\right)^{\beta'}\left(\dfrac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta'+\gamma+\alpha',\ \beta'+\gamma'+\alpha,\ 1+\beta'-\beta,\ \dfrac{(a-c)(z-b)}{(a-b)(z-c)}\right\}, \end{cases}$$

$$\begin{cases} y_9 = \left(\dfrac{z-c}{z-a}\right)^{\gamma}\left(\dfrac{z-b}{z-a}\right)^{\beta} F\left\{\gamma+\alpha+\beta,\ \gamma+\alpha'+\beta',\ 1+\gamma-\gamma',\ \dfrac{(b-a)(z-c)}{(b-c)(z-a)}\right\}, \\ y_{10} = \left(\dfrac{z-c}{z-a}\right)^{\gamma'}\left(\dfrac{z-b}{z-a}\right)^{\beta} F\left\{\gamma'+\alpha+\beta,\ \gamma'+\alpha'+\beta',\ 1+\gamma'-\gamma,\ \dfrac{(b-a)(z-c)}{(b-c)(z-a)}\right\}, \\ y_{11} = \left(\dfrac{z-c}{z-a}\right)^{\gamma}\left(\dfrac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma+\alpha+\beta',\ \gamma+\alpha'+\beta,\ 1+\gamma-\gamma',\ \dfrac{(b-a)(z-c)}{(b-c)(z-a)}\right\}, \\ y_{12} = \left(\dfrac{z-c}{z-a}\right)^{\gamma'}\left(\dfrac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma'+\alpha+\beta',\ \gamma'+\alpha'+\beta,\ 1+\gamma'-\gamma,\ \dfrac{(b-a)(z-c)}{(b-c)(z-a)}\right\}, \end{cases}$$

$$\begin{cases} y_{13} = \left(\dfrac{z-a}{z-c}\right)^{\alpha}\left(\dfrac{z-b}{z-c}\right)^{\beta} F\left\{\alpha+\gamma+\beta,\ \alpha+\gamma'+\beta',\ 1+\alpha-\alpha',\ \dfrac{(b-c)(z-a)}{(b-a)(z-c)}\right\}, \\ y_{14} = \left(\dfrac{z-a}{z-c}\right)^{\alpha'}\left(\dfrac{z-b}{z-c}\right)^{\beta} F\left\{\alpha'+\gamma+\beta,\ \alpha'+\gamma'+\beta',\ 1+\alpha'-\alpha,\ \dfrac{(b-c)(z-a)}{(b-a)(z-c)}\right\}, \\ y_{15} = \left(\dfrac{z-a}{z-c}\right)^{\alpha}\left(\dfrac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha+\gamma+\beta',\ \alpha+\gamma'+\beta,\ 1+\alpha-\alpha',\ \dfrac{(b-c)(z-a)}{(b-a)(z-c)}\right\}, \\ y_{16} = \left(\dfrac{z-a}{z-c}\right)^{\alpha'}\left(\dfrac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha'+\gamma+\beta',\ \alpha'+\gamma'+\beta,\ 1+\alpha'-\alpha,\ \dfrac{(b-c)(z-a)}{(b-a)(z-c)}\right\}, \end{cases}$$

$$\begin{cases} y_{17} = \left(\dfrac{z-c}{z-b}\right)^{\gamma}\left(\dfrac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma+\beta+\alpha,\ \gamma+\beta'+\alpha',\ 1+\gamma-\gamma',\ \dfrac{(a-b)(z-c)}{(a-c)(z-b)}\right\}, \\ y_{18} = \left(\dfrac{z-c}{z-b}\right)^{\gamma'}\left(\dfrac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma'+\beta+\alpha,\ \gamma'+\beta'+\alpha',\ 1+\gamma'-\gamma,\ \dfrac{(a-b)(z-c)}{(a-c)(z-b)}\right\}, \\ y_{19} = \left(\dfrac{z-c}{z-b}\right)^{\gamma}\left(\dfrac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma+\beta+\alpha',\ \gamma+\beta'+\alpha,\ 1+\gamma-\gamma',\ \dfrac{(a-b)(z-c)}{(a-c)(z-b)}\right\}, \\ y_{20} = \left(\dfrac{z-c}{z-b}\right)^{\gamma'}\left(\dfrac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma'+\beta+\alpha',\ \gamma'+\beta'+\alpha,\ 1+\gamma'-\gamma,\ \dfrac{(a-b)(z-c)}{(a-c)(z-b)}\right\}, \end{cases}$$

$$\begin{cases} y_{21} = \left(\dfrac{z-b}{z-a}\right)^{\beta} \left(\dfrac{z-c}{z-a}\right)^{\gamma} F\left\{\beta+\alpha+\gamma,\ \beta+\alpha'+\gamma',\ 1+\beta-\beta',\ \dfrac{(c-a)(z-b)}{(c-b)(z-a)}\right\}, \\ y_{22} = \left(\dfrac{z-b}{z-a}\right)^{\beta'} \left(\dfrac{z-c}{z-a}\right)^{\gamma} F\left\{\beta'+\alpha+\gamma,\ \beta'+\alpha'+\gamma',\ 1+\beta'-\beta,\ \dfrac{(c-a)(z-b)}{(c-b)(z-a)}\right\}, \\ y_{23} = \left(\dfrac{z-b}{z-a}\right)^{\beta} \left(\dfrac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta+\alpha+\gamma',\ \beta+\alpha'+\gamma,\ 1+\beta-\beta',\ \dfrac{(c-a)(z-b)}{(c-b)(z-a)}\right\}, \\ y_{24} = \left(\dfrac{z-b}{z-a}\right)^{\beta'} \left(\dfrac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta'+\alpha+\gamma',\ \beta'+\alpha'+\gamma,\ 1+\beta'-\beta,\ \dfrac{(c-a)(z-b)}{(c-b)(z-a)}\right\}. \end{cases}$$

The existence of these twenty-four values was first shewn by Kummer*.

Example. Find the twenty-four solutions of the Legendre differential equation, corresponding to the above set of solutions of the hypergeometric differential equation; and express each of them in terms of the two independent solutions $P_n(z)$ and $Q_n(z)$.

141. *Relations between the particular solutions of the hypergeometric differential equation.*

Since the twenty-four expressions found in the last article are solutions of the same linear differential equation of the second order, any three of them must be connected by a linear relation with constant coefficients.

We proceed to find the relations which thus connect them.

First, consider the set of four solutions

$$y_1,\quad y_3,\quad y_{13},\quad y_{15};$$

it is clear that, in the neighbourhood of the point $z=a$, each of them can be expanded in a power-series of the form

$$A\,(z-a)^{\alpha}\,\{1+B\,(z-a)+C\,(z-a)^2+\ldots\}.$$

But there is only one series of the form

$$(z-a)^{\alpha}\,\{1+B\,(z-a)+C\,(z-a)^2+\ldots\}$$

which satisfies the differential equation; for the coefficients $B, C, \ldots$ can be uniquely determined by actual substitution in the differential equation. Let this solution be denoted by $P^{(\alpha)}$.

Thus the solutions

$$y_1,\quad y_3,\quad y_{13},\quad y_{15}$$

must be mere multiples of $P^{(\alpha)}$. Moreover,

for y_1 the factor A is $\quad (a-c)^{\gamma}\,(a-b)^{-(\alpha+\gamma)}$;

for y_3 it is $\quad (a-c)^{\gamma'}(a-b)^{-(\alpha+\gamma')}$;

for y_{13} it is $\quad (a-b)^{\beta}\,(a-c)^{-\alpha-\beta}$;

and for y_{15} it is $\quad (a-b)^{\beta'}(a-c)^{-\alpha-\beta'}$.

* *Crelle's Journal*, xv.

Thus we have

$$P^{(\alpha)} = (z-a)^\alpha \left(\frac{z-c}{a-c}\right)^\gamma \left(\frac{z-b}{a-b}\right)^{-\alpha-\gamma}$$

$$\times F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma',\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

$$= (z-a)^\alpha \left(\frac{z-c}{a-c}\right)^{\gamma'} \left(\frac{z-b}{a-b}\right)^{-\alpha-\gamma'}$$

$$\times F\left\{\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

$$= (z-a)^\alpha \left(\frac{z-c}{a-c}\right)^{-\alpha-\beta} \left(\frac{z-b}{a-b}\right)^{\beta}$$

$$\times F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma',\ 1+\alpha-\alpha',\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

$$= (z-a)^\alpha \left(\frac{z-c}{a-c}\right)^{-\alpha-\beta'} \left(\frac{z-b}{a-b}\right)^{\beta'}$$

$$\times F\left\{\alpha+\beta'+\gamma,\ \alpha+\beta+\gamma',\ 1+\alpha-\alpha',\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}.$$

Similarly solutions $P^{(\alpha')}$, $P^{(\beta)}$, $P^{(\beta')}$, $P^{(\gamma)}$, $P^{(\gamma')}$ exist, each of which is equivalent to four of the above hypergeometric series.

Having thus classified the twenty-four solutions into six distinct solutions, namely

$$P^{(\alpha)},\ P^{(\alpha')},\ P^{(\beta)},\ P^{(\beta')},\ P^{(\gamma)},\ P^{(\gamma')},$$

we proceed to find the relations between these latter six solutions. We know that $P^{(\alpha)}$ must be expressible linearly in terms of $P^{(\gamma)}$ and $P^{(\gamma')}$. Let the relation between them be

$$P^{(\alpha)} = \alpha_\gamma P^{(\gamma)} + \alpha_{\gamma'} P^{(\gamma')}.$$

We have then to find the coefficients α_γ and $\alpha_{\gamma'}$.

Now this equation can be written in the form

$$(z-a)^\alpha \left(\frac{z-c}{a-c}\right)^\gamma \left(\frac{z-b}{a-b}\right)^{-\alpha-\gamma}$$

$$\times F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma',\ 1+\alpha-\alpha',\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

$$= \alpha_\gamma (z-c)^\gamma \left(\frac{z-a}{c-a}\right)^\alpha \left(\frac{z-b}{c-b}\right)^{-\alpha-\gamma}$$

$$\times F\left\{\alpha+\beta+\gamma,\ \alpha'+\beta'+\gamma,\ 1+\gamma-\gamma',\ \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

$$+ \alpha_{\gamma'} (z-c)^{\gamma'} \left(\frac{z-a}{c-a}\right)^\alpha \left(\frac{z-b}{c-b}\right)^{-\alpha-\gamma'}$$

$$\times F\left\{\alpha+\beta+\gamma',\ \alpha'+\beta'+\gamma',\ 1+\gamma'-\gamma,\ \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}.$$

Dividing throughout by the common factor $(z-a)^a$, and writing $z=a$ and $z=c$ successively in the resulting equation, we obtain two equations, from which σ_γ and $\alpha_{\gamma'}$ can be found: the hypergeometric functions reduce to the type

$$F(u, v, w, 1),$$

which in § 135 was shewn to be expressible in terms of Gamma-functions, and the type $F(u, v, w, 0)$, which clearly has the value unity.

As already explained, in certain cases (e.g. when one of the exponent-differences is an integer) the above theory of the solutions requires modification. For a discussion of these cases the student is referred to Lindelöf's paper already mentioned, and Klein's Lectures "Ueber die hypergeometrische Function."

142. *Solution of the general hypergeometric differential equation by a definite integral.*

We next proceed to establish a result of great importance, relating to the expression of the hypergeometric function by means of definite integrals.

Let the dependent variable y in the differential equation of the general hypergeometric function ((A) of § 137) be replaced by a new dependent variable I, defined by the relation

$$y=(z-a)^\alpha (z-b)^\beta (z-c)^\gamma I.$$

The differential equation satisfied by I is easily found to be

$$0=\frac{d^2I}{dz^2}+\left\{\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}\right\}\frac{dI}{dz}$$
$$+\frac{(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma+1)z+\Sigma a(\alpha+\beta'+\gamma'-1)\}}{(z-a)(z-b)(z-c)}I,$$

which can be written in the form

$$0=Q(z)\frac{d^2I}{dz^2}-\{(\xi-2)Q'(z)+R(z)\}\frac{dI}{dz}$$
$$+\left\{\frac{(\xi-2)(\xi-1)}{2}Q''(z)+(\xi-1)R'(z)\right\}I,$$

where

$$\begin{cases} \xi=1-\alpha-\beta-\gamma=\alpha'+\beta'+\gamma', \\ Q(z)=(z-a)(z-b)(z-c), \\ R(z)=\Sigma(\alpha'+\beta+\gamma)(z-b)(z-c). \end{cases}$$

It must be observed that the function I is not regular at $z=\infty$, and consequently the above differential equation in I is not a case of the generalised hypergeometric equation.

We shall now shew that this differential equation can be satisfied by an integral of the form

$$I = \int_C (t-a)^{\alpha'+\beta+\gamma-1} (t-b)^{\alpha+\beta'+\gamma-1} (t-c)^{\alpha+\beta+\gamma'-1} (z-t)^{-\alpha-\beta-\gamma}\, dt,$$

provided the path C of integration is suitably chosen.

For on substituting this value of I in the differential equation, the condition that the equation should be satisfied becomes

$$0 = \int_C (t-a)^{\alpha'+\beta+\gamma-1} (t-b)^{\alpha+\beta'+\gamma-1} (t-c)^{\alpha+\beta+\gamma'-1} (z-t)^{-\alpha-\beta-\gamma-2}\, K dt,$$

where

$$K = (\xi - 2) \left\{ Q(z) + (t-z) Q'(z) + \frac{(t-z)^2}{2} Q''(z) \right\}$$
$$+ (t-z) \{R(z) + (t-z) R'(z)\}$$
$$= (\xi - 2) \{Q(t) - (t-z)^3\} + (t-z) \{R(t) - (t-z)^2 \Sigma (\alpha' + \beta + \gamma)\}$$
$$= (\xi - 2) Q(t) + (t-z) R(t)$$
$$= -(1 + \alpha + \beta + \gamma)(t-a)(t-b)(t-c)$$
$$+ \Sigma (\alpha' + \beta + \gamma)(t-b)(t-c)(t-z),$$

or $K = (t-a)^{1-\alpha'-\beta-\gamma} (t-b)^{1-\alpha-\beta'-\gamma} (t-c)^{1-\alpha-\beta-\gamma'} (z-t)^{\alpha+\beta+\gamma+2}$

$$\frac{d}{dt} \{(t-a)^{\alpha'+\beta+\gamma} (t-b)^{\alpha+\beta'+\gamma} (t-c)^{\alpha+\beta+\gamma'} (t-z)^{-(1+\alpha+\beta+\gamma)}\}.$$

It follows that the condition to be satisfied reduces to

$$0 = \int_C \frac{dV}{dt} dt = \int_C dV,$$

where $V = (t-a)^{\alpha'+\beta+\gamma} (t-b)^{\alpha+\beta'+\gamma} (t-c)^{\alpha+\beta+\gamma'} (t-z)^{-(1+\alpha+\beta+\gamma)}$.

The integral I will therefore be a solution of the differential equation, provided the path of integration C is such that the quantity V resumes its initial value after describing the arc C.

Now $V = (t-a)^{\alpha'+\beta+\gamma-1} (t-b)^{\alpha+\beta'+\gamma-1} (t-c)^{\alpha+\beta+\gamma'-1} (z-t)^{-\alpha-\beta-\gamma}\, U,$

where $U = (t-a)(t-b)(t-c)(z-t)^{-1}$;

and the quantity U resumes its original value after describing any contour: hence if C be a closed contour, it must be such that the integrand in the integral I resumes its original value after describing the contour.

Hence finally *any integral of the type*

$$(z-a)^{\alpha} (z-b)^{\beta} (z-c)^{\gamma} \int_C (t-a)^{\beta+\gamma+\alpha'-1} (t-b)^{\gamma+\alpha+\beta'-1} (t-c)^{\alpha+\beta+\gamma'-1} (z-t)^{-\alpha-\beta-\gamma} dt,$$

where C is either a closed contour in the t-plane such that the integrand resumes its initial value after describing it, or else is an arc such that the quantity V has the same value at its termini, is a solution of the differential equation of the general hypergeometric function.

Example 1. As an example, we shall now deduce a real definite integral which (for a certain range of values of the quantities involved) represents the hypergeometric series.

The hypergeometric series $F(a, b, c, z)$ is, as already shewn, a solution of the differential equation of the function

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}.$$

The integral

$$(z-a)^\alpha\left(1-\frac{z}{b}\right)^\beta(z-c)^\gamma\int_C(t-a)^{\beta+\gamma+\alpha'-1}\left(1-\frac{t}{b}\right)^{\gamma+\alpha+\beta'-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\,dt$$

thus becomes in this case

$$\int_C t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt.$$

Now the quantity V is in this case

$$t^{1-c+a}(t-1)^{c-b}(t-z)^{-1-a},$$

and this tends to zero at $t=1$ and $t=\infty$, provided $c>b>0$.

Hence if these conditions are fulfilled, we can take as the contour C an arc in the t-plane joining the points $t=1$ and $t=\infty$; so that a solution of the differential equation is

$$\int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt.$$

In this integral, write $t=\frac{1}{u}$; the integral becomes

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}\,du;$$

this integral is therefore a solution of the differential equation for the hypergeometric series.

It is easily seen that this integral is in fact a mere multiple of the hypergeometric series

$$F(a, b, c, z);$$

for supposing $|z|<1$, and expanding the quantity $(1-uz)^{-a}$ in ascending powers of z by the Binomial Theorem, the integral takes the form

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}\,du+\sum_{r=1}^{\infty}\frac{a(a+1)\dots(a+r-1)}{r!}z^r\int_0^1 u^{b-1+r}(1-u)^{c-b-1}\,du,$$

or

$$B(b, c-b)+\sum_{r=1}^{\infty}\frac{a(a+1)\dots(a+r-1)}{r!}B(b+r, c-b)\,z^r,$$

or

$$B(b, c-b)\left\{1+\sum_{r=1}^{\infty}\frac{a(a+1)\dots(a+r-1)\,b(b+1)\dots(b+r-1)}{r!\,c(c+1)\dots(c+r-1)}z^r\right\},$$

or

$$B(b, c-b)\,F(a, b, c, z),$$

which establishes the result stated.

Example 2. *Deduce Schläfli's integral for the Legendre functions, as a case of the general hypergeometric integral.*

Since the Legendre equation corresponds to the hypergeometric function

$$P\left\{\begin{matrix} -1 & \infty & 1 & \\ 0 & n+1 & 0 & z \\ 0 & -n & 0 & \end{matrix}\right\},$$

the corresponding integral is

$$\left(1-\frac{z}{\infty}\right)^{n+1}\int_C (t+1)^n(t-1)^n\left(1-\frac{t}{\infty}\right)^{-n}(t-z)^{-n-1}\,dt,$$

or

$$\int_C (t^2-1)^n(t-z)^{-n-1}\,dt,$$

taken round a contour C such that the integrand resumes its initial value after describing it; and this is Schläfli's integral.

Example 3. *Deduce Laplace's integral for the Legendre functions, as a case of the general hypergeometric integral.*

If we write

$$z=\tfrac{1}{2}(\xi^{\frac{1}{2}}+\xi^{-\frac{1}{2}}),$$

the Legendre differential equation becomes

$$\frac{d^2y}{d\xi^2}+\left(\frac{1}{2\xi}+\frac{1}{\xi-1}\right)\frac{dy}{d\xi}-\frac{n(n+1)}{4}\frac{y}{\xi^2}=0.$$

This corresponds to the hypergeometric function

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ -\frac{n}{2} & \frac{n+1}{2} & 0 & \xi \\ \frac{n+1}{2} & -\frac{n}{2} & 0 & \end{matrix}\right\},$$

and so the hypergeometric integral becomes in this case

$$\xi^{-\frac{n}{2}}\int u^n(1-u)^{-\frac{1}{2}}(\xi-u)^{-\frac{1}{2}}\,du,$$

taken round a contour enclosing the points $u=1$ and $u=\xi$.

Write

$$\xi=\zeta^2,$$

so

$$\zeta=z+(z^2-1)^{\frac{1}{2}}.$$

Then the integral becomes

$$\zeta^{-n-1}\int(1-u)^{-\frac{1}{2}}(1-\zeta^{-2}u)^{-\frac{1}{2}}u^n\,du,$$

taken round a contour enclosing the points $u=1$ and $u=\zeta^2$.

Write $u=h\zeta$ in this integral; we thus obtain

$$\int (1-2zh+h^2)^{-\frac{1}{2}} h^n dh,$$

the integral being now taken round a contour in the h-plane enclosing the points $h=\zeta$ and $h=\zeta^{-1}$.

Suppose now that the real part of z is positive; and let the contour become so attenuated as to reduce to a small circle surrounding the point $h=\zeta$, another small circle surrounding the point $h=\zeta^{-1}$, and the line joining the points ζ and ζ^{-1}, described twice. The small circles contribute only infinitesimally to the integral, which thus becomes a multiple of

$$\int_{\zeta^{-1}}^{\zeta} (1-2zh+h^2)^{-\frac{1}{2}} h^n dh.$$

Writing
$$h=z+(z^2-1)^{\frac{1}{2}}\cos\phi$$

in this integral, we obtain

$$\int_0^{\pi} \{z+(z^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi,$$

which is one of Laplace's integrals (§ 119).

143. *Determination of the integral which represents $P^{(\alpha)}$.*

We shall now shew how the integral which represents the particular solution $P^{(\alpha)}$ (§ 141) of the hypergeometric differential equation can be found.

We have seen (§ 142) that the integral

$$I=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\gamma+\alpha+\beta'-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}dt$$

satisfies the differential equation of the hypergeometric function, provided C is a closed contour such that the integrand resumes its initial value after describing C. Now the singularities of this integrand in the t-plane are the points a, b, c, z; and on describing a simple closed contour enclosing the singularity b alone, the integrand resumes its initial value multiplied by

$$e^{2\pi i(\gamma+\alpha+\beta'-1)},$$

as is seen by writing it in the form

$$e^{(\beta+\gamma+\alpha'-1)\log(t-a)+(\gamma+\alpha+\beta'-1)\log(t-b)+(\alpha+\beta+\gamma'-1)\log(t-c)-(\alpha+\beta+\gamma)\log(z-t)}.$$

Take then a point O in the t-plane, and draw a loop in the t-plane passing through O and encircling the point b, but not encircling any of the points a, c, z. Let an integral taken in the positive or counter-clockwise direction of circulation round the perimeter of this loop be denoted by the sign

$$\int_0^{(b+)};$$

and let an integral taken in the negative direction of circulation round the perimeter of the loop be denoted by

$$\int_0^{(b-)},$$

so that we have the equation

$$\int_0^{(b+)} = -\int_0^{(b-)},$$

where it is understood that the initial value of the integrand in the second integral is taken equal to the final value of the integrand in the first integral.

Let now a contour C be drawn in the following way. Take first a loop starting from O, encircling the point b in the positive direction, and returning to O; then a loop starting from O, encircling the point c in the positive direction, and returning to O; then a loop encircling the point b in the negative direction; and lastly a loop encircling the point c in the negative or clockwise direction.

Conformably to the notation already explained, an integral taken round this contour will be denoted by

$$\int_0^{(b+,\, c+,\, b-,\, c-)}.$$

Now after description of this contour, the integrand of the integral I already considered resumes its initial value multiplied by

$$e^{2\pi i\,(\gamma+\alpha+\beta'-1+\alpha+\beta+\gamma'-1-\gamma-\alpha-\beta'+1-\alpha-\beta-\gamma'+1)},$$

or 1, i.e. the integrand resumes its initial value*.

Hence if C be taken as the contour, the integral I will satisfy the differential equation.

Thus

$$I = (z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_0^{(b+,\, c+,\, b-,\, c-)}(t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\gamma+\alpha+\beta'-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}\,dt$$

satisfies the differential equation of the hypergeometric function.

Now suppose that the point z is taken near to the point a, so that $|z-a|$ is less than either $|b-a|$ or $|c-a|$. We can clearly draw the contour just

* These *double-circuit integrals* were introduced by Jordan in 1887. Clearly any number of contours can be formed in this way, it being necessary only to ensure that each singular point is encircled as often in the negative or clockwise direction of circulation as in the positive or counter-clockwise direction.

described in such a way that, for all points t on it, $|t-a|$ is greater than $|z-a|$. Thus we can write

$$I=(z-a)^{\alpha}(a-b)^{\beta}\left(1-\frac{z-a}{b-a}\right)^{\beta}(a-c)^{\gamma}\left(1-\frac{z-a}{c-a}\right)^{\gamma}$$

$$\times\int_0^{(b+,c+,b-,c-)}(t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\gamma+\alpha+\beta'-1}(t-c)^{\alpha+\beta+\gamma'-1}$$

$$(a-t)^{-\alpha-\beta-\gamma}\left(1-\frac{z-a}{t-a}\right)^{-\alpha-\beta-\gamma}dt.$$

Under the conditions already stated, each of the expressions

$$\left(1-\frac{z-a}{b-a}\right)^{\beta},\quad\left(1-\frac{z-a}{c-a}\right)^{\gamma},\quad\text{and}\quad\left(1-\frac{z-a}{t-a}\right)^{-\alpha-\beta-\gamma},$$

can be expanded by the Binomial Theorem in ascending powers of $(z-a)$. We thus obtain for I an expansion of the form

$$I=(z-a)^{\alpha}\{A+B(z-a)+C(z-a)^2+\dots\},$$

and as I satisfies the differential equation it must therefore be a multiple of the particular solution $P^{(\alpha)}$ of § 141.

Thus

$$P^{(\alpha)}=\text{Constant}\times(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_0^{(b+,c+,b-,c-)}(t-a)^{\beta+\gamma+\alpha'-1}$$

$$(t-b)^{\gamma+\alpha+\beta'-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}dt.$$

Similarly

$$P^{(\alpha')}=\text{Constant}\times(z-a)^{\alpha'}(z-b)^{\beta}(z-c)^{\gamma}\int_0^{(b+,c+,b-,c-)}(t-a)^{\beta+\gamma+\alpha-1}$$

$$(t-b)^{\gamma+\alpha'+\beta'-1}(t-c)^{\alpha'+\beta+\gamma'-1}(z-t)^{-\alpha'-\beta-\gamma}dt.$$

In the same way the particular solutions $P^{(\beta)}$, $P^{(\beta')}$, $P^{(\gamma)}$, $P^{(\gamma')}$, can be expressed as contour-integrals.

144. *Evaluation of a double-contour integral.*

We may note that an integral

$$\int_0^{(a+,b+,a-,b-)}$$

can be expressed in terms of the integrals

$$\int_0^{(a+)}\text{ and }\int_0^{(b+)},$$

in the following way.

Let the initial value of the integrand at the point O be denoted by T. After describing the loop round a, the integrand will have at O the value $e^{2\pi i(\alpha'+\beta+\gamma-1)}T$, and the part

$\int_0^{(a+)}$ of the integral $\int_0^{(a+, b+, a-, b-)}$ will have been obtained. Describing next the loop round b, the corresponding part of the integral $\int_0^{(a+, b+, a-, b-)}$ will therefore be

$$e^{2\pi i(\alpha+\beta'+\gamma-1)}\int_0^{(b+)},$$

and the integrand will return to O with the value

$$e^{2\pi i(\alpha'+\beta+\gamma-1+\alpha+\beta'+\gamma-1)}\,T.$$

Describing next the loop round a in the negative direction, we observe that the corresponding part of the integral would have been

$$\int_0^{(a-)}$$

if the integrand had had for initial value

$$e^{2\pi i(\alpha'+\beta+\gamma-1)}\,T,$$

which is its final value when the loop is described with the initial value T: it is therefore actually

$$e^{2\pi i(\alpha+\beta'+\gamma-1)}\int_0^{(a-)},$$

or

$$-e^{2\pi i(\alpha+\beta'+\gamma-1)}\int_0^{(a+)};$$

and lastly, describing the loop round b in the negative direction, we obtain the part

$$-\int_0^{(b+)}$$

of the integral.

Collecting these results, we have

$$\int_0^{(a+, b+, a-, b-)} = \left(1-e^{2\pi i(\alpha+\beta'+\gamma-1)}\right)\int_0^{(a+)} - \left(1-e^{2\pi i(\alpha+\beta'+\gamma-1)}\right)\int_0^{(b+)},$$

a formula which furnishes the value of the double-contour integral in terms of two simple-contour integrals.

145. *Relations between contiguous hypergeometric functions.*

Let $P(z)$ be a hypergeometric function with the argument z, the singularities a, b, c, and the exponents $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$. Let $P_{l+1, m-1}(z)$ denote the function which is obtained by replacing two of the exponents, l and m, in $P(z)$ by $l+1$ and $m-1$ respectively. Such functions $P_{l+1, m-1}(z)$ are said to be *contiguous* to $P(z)$. There are clearly 6×5 or 30 contiguous functions, since l and m may be any two of the six exponents.

It was first shewn by Riemann* that *the function $P(z)$ and any two of its contiguous functions are connected by a linear relation, the coefficients in which are polynomials in z.*

* *Abhandlungen der Kön. Ges. der Wiss. zu Göttingen*, 1857.

There will clearly be $\frac{30.29}{2}$ or 435 of these relations. In order to obtain them, we shall take $P(z)$ in the form

$$P(z) = (z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\gamma+\alpha+\beta'-1}$$
$$(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\,dt,$$

where C may be any closed contour in the t-plane such that the integrand resumes its initial value after describing C.

First, since the integral round C of the differential of any function which resumes its initial value after describing C is zero, we have

$$0 = \int_C \frac{d}{dt}\{(t-a)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\}\,dt,$$

or

$$0 = (\alpha'+\beta+\gamma)\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\,dt$$
$$+(\alpha+\beta'+\gamma-1)\int_C (t-a)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma-2}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\,dt$$
$$+(\alpha+\beta+\gamma'-1)\int_C (t-a)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-2}(t-z)^{-\alpha-\beta-\gamma}\,dt$$
$$-(\alpha+\beta+\gamma)\int_C (t-a)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma-1}\,dt,$$

or

$$(\alpha'+\beta+\gamma)P + (\alpha+\beta'+\gamma-1)P_{\alpha'+1,\beta'-1} + (\alpha+\beta+\gamma'-1)P_{\alpha'+1,\gamma'-1}$$
$$= \frac{(\alpha+\beta+\gamma)}{z-b}P_{\beta+1,\gamma'-1}.$$

Considerations of symmetry shew that the right-hand side of this equation can be replaced by

$$\frac{(\alpha+\beta+\gamma)}{z-c}P_{\beta'-1,\gamma+1}.$$

These, together with the analogous formulae obtained by cyclical interchange of (a, α, α') with (b, β, β') and (c, γ, γ'), are six linear relations connecting the hypergeometric function P with the twelve contiguous functions

$$P_{\alpha+1,\beta'-1},\ P_{\beta+1,\gamma'-1},\ P_{\gamma+1,\alpha'-1},\ P_{\alpha+1,\gamma'-1},\ P_{\beta+1,\alpha'-1},\ P_{\gamma+1,\beta'-1},$$
$$P_{\alpha'+1,\beta'-1},\ P_{\alpha'+1,\gamma'-1},\ P_{\beta'+1,\gamma'-1},\ P_{\beta'+1,\alpha'-1},\ P_{\gamma'+1,\alpha'-1},\ P_{\gamma'+1,\beta'-1}.$$

Next, writing $t-a=(t-b)+(b-a)$, and using $P_{\alpha'-1}$ to denote the result of writing $\alpha'-1$ for α' in P, we have

$$P = P_{\alpha'-1,\,\beta'+1} + (b-a)\,P_{\alpha'-1}.$$

Similarly
$$P = P_{\alpha'-1,\,\gamma'+1} + (c-a)\,P_{\alpha'-1}.$$

Eliminating $P_{\alpha'-1}$ from these equations, we have

$$(c-b)\,P + (a-c)\,P_{\alpha'-1,\,\beta'+1} + (b-a)\,P_{\alpha'-1,\,\gamma'+1} = 0.$$

This and the analogous formulae are three more linear relations connecting P with the last six of the twelve contiguous functions written above.

Next, writing $(t-z)=(t-a)-(z-a)$ we readily find the relation

$$P = \frac{1}{z-b}\,P_{\beta+1,\gamma'-1} - (z-a)^{\alpha+1}(z-b)^{\beta}(z-c)^{\gamma}$$
$$\times \int_C (t-a)^{\beta+\gamma+\alpha'-1}(z-a)^{\gamma+\alpha+\beta'-1}(z-b)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma-1}\,dt,$$

which gives the equations

$$(z-a)^{-1}\{P-(z-b)^{-1}P_{\beta+1,\gamma'-1}\} = (z-b)^{-1}\{P-(z-c)^{-1}P_{\gamma+1,\alpha'-1}\}$$
$$= (z-c)^{-1}\{P-(z-a)^{-1}P_{\alpha+1,\beta'-1}\}.$$

These are two more linear equations between P and the above twelve contiguous functions.

We have therefore now altogether found eleven linear relations between P and these twelve functions, the coefficients in these relations being rational functions of z. Hence each of these functions can be expressed linearly in terms of P and some selected one of them; that is, *between P and any two of the above functions there exists a linear relation.* The coefficients in this relation will be rational functions of z, and therefore will become polynomials in z when the relation is multiplied throughout by the least common multiple of their denominators.

The theorem is therefore proved, so far as the above twelve contiguous functions are concerned. It can in the same way be extended so as to be established for the rest of the thirty contiguous functions.

Corollary. If functions be derived from P by replacing the exponents $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$, by $\alpha+p$, $\alpha'+q$, $\beta+r$, $\beta'+s$, $\gamma+t$, $\gamma'+u$, where p, q, r, s, t, u, are integers satisfying the relation

$$p+q+r+s+t+u=0,$$

then between P and these functions there exists a linear relation, the coefficients in which are polynomials in z.

This result can be obtained by connecting P with the two functions by a chain of intermediate contiguous functions, writing down the linear relations which connect them with P and the two functions, and from these relations eliminating the intermediate contiguous functions.

It will be noticed that many of the theorems found elsewhere in this book, e.g. the recurrence-formulae for the Legendre functions (§ 117), are really cases of the theorem of this article.

MISCELLANEOUS EXAMPLES.

1. Shew that

$$F(a, b+1, c, z) - F(a, b, c, z) = \frac{az}{c} F(a+1, b+1, c+1, z).$$

2. Shew that

$$F(a+1, b+1, c, z) - F(a, b, c, z) = \frac{abz}{c^2} F(a+1, b+1, c+1, z).$$

3. If $P(z)$ be a hypergeometric function, express its derivates $\frac{dP}{dz}$ and $\frac{d^2P}{dz^2}$ linearly in terms of P and contiguous functions, and hence find the linear relation between P, $\frac{dP}{dz}$, and $\frac{d^2P}{dz^2}$, i.e. verify that P satisfies the hypergeometric differential equation.

4. If

$$W(a, b, x) \text{ denote } F\left(\frac{b-a}{b},\ 1,\ 2,\ -bx\right),$$

shew that the equation $\quad y = W(a, b, x)$

is equivalent to $\quad x = W(b, a, y)$.

5. Shew that a second solution of the differential equation for

$$F(a, b, c, x)$$

is

$$x^{1-c} F(a-c+1,\ \ b-c+1,\ \ 2-c,\ \ x).$$

6. Shew that the equation

$$(a_2 + b_2 x)\frac{d^2y}{dx^2} + (a_1 + b_1 x)\frac{dy}{dx} + (a_0 + b_0 x)\, y = 0$$

can, by change of variables, be brought to the form

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0,$$

and that this latter equation can be derived from the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + \{c - (a+b+1)x\}\frac{dy}{dx} - aby = 0$$

by the substitution $b = m$, $x = \frac{x'}{m}$, where m is infinitely large.

7. Shew that

$$C_n^{\nu}(z) = P\begin{Bmatrix} -1 & \infty & 1 & \\ \frac{1}{2}-\nu & n+2\nu & \frac{1}{2}-\nu & z \\ 0 & -n & 0 & \end{Bmatrix},$$

where $C^{\nu}(z)$ is the coefficient of h^n in the expansion of $(1-2hz+h^2)^{-\nu}$ in ascending powers of h.

8. Shew that, for values of x between 0 and 1, the solution of the equation

$$x(1-x)\frac{d^2y}{dx^2} + \frac{1}{2}(\alpha+\beta+1)(1-2x)\frac{dy}{dx} - \alpha\beta y = 0$$

is $$AF\{\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\beta,\ \tfrac{1}{2},\ (1-2x)^2\} + B(1-2x)F\{\tfrac{1}{2}(\alpha+1),\ \tfrac{1}{2}(\beta+1),\ \tfrac{3}{2},\ (1-2x)^2\},$$

where A, B, are arbitrary constants and $F(\alpha, \beta, \gamma, x)$ represents the hypergeometric series.

(Cambridge Mathematical Tripos, Part I, 1896.)

9. Shew that the differential equation for the associated Legendre function $P_n^m(z)$ of order n and degree m is satisfied by the three functions

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \frac{1}{2}m & -n & \frac{1}{2}m & \frac{1-z}{2} \\ -\frac{1}{2}m & n+1 & -\frac{1}{2}m & \end{Bmatrix},$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ -\frac{n}{2} & m & -\frac{n}{2} & \frac{z+(z^2-1)^{\frac{1}{2}}}{z-(z^2-1)^{\frac{1}{2}}} \\ \frac{n+1}{2} & -m & \frac{n+1}{2} & \end{Bmatrix},$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ -\frac{n}{2} & \frac{m}{2} & 0 & \frac{1}{1-z^2} \\ \frac{n+1}{2} & -\frac{m}{2} & \frac{1}{2} & \end{Bmatrix}.$$

(Olbricht.)

10. Shew that the hypergeometric equation

$$x(x-1)\frac{d^2y}{dx^2} - \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} + \alpha\beta y = 0$$

is satisfied by the two integrals

$$\int_0^1 z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dz$$

and $$\int_0^1 z^{\beta-1}(1-z)^{\alpha-\gamma}\{1-(1-x)z\}^{-\alpha}\,dz.$$

11. If

$$(1-x)^{\alpha+\beta-\gamma} F(2\alpha, 2\beta, 2\gamma, x) = 1 + Bx + Cx^2 + Dx^3 + \dots,$$

shew that

$$F(\alpha, \beta, \gamma+\tfrac{1}{2}, x) F(\gamma-\alpha, \gamma-\beta, \gamma+\tfrac{1}{2}, x)$$

$$= 1 + \frac{\gamma}{\gamma+\frac{1}{2}} Bx + \frac{\gamma \cdot \gamma+1}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})} Cx^2 + \frac{\gamma(\gamma+1)(\gamma+2)}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})(\gamma+\frac{5}{2})} Dx^3 + \dots.$$

(Cayley.)

12. Prove that

$$P_n(z) = \frac{1}{\pi} \tan n\pi \{Q_n(z) - Q_{-n-1}(z)\},$$

where $P_n(z)$ and $Q_n(z)$ are the Legendre functions of the first and second kind of order n.

13. If a function $F(\alpha, \beta, \beta', \gamma;\ x, y)$ be defined by the equation

$$F(\alpha, \beta, \beta', \gamma;\ x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\,du,$$

then shew that between F and any three of its eight contiguous functions

$$F(\alpha \pm 1), \quad F(\beta \pm 1), \quad F(\beta' \pm 1), \quad F(\gamma \pm 1),$$

there exists a homogeneous linear equation, whose coefficients are polynomials in x and y.

(Levavasseur.)

14. If $\gamma-\alpha-\beta < 0$, shew that, for values of x nearly equal to unity,

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-x)^{\gamma-\alpha-\beta},$$

and that if $\gamma-\alpha-\beta=0$, the corresponding approximate formula is

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log \frac{1}{1-x}.$$

(Cambridge Mathematical Tripos, Part II, 1893.)

15. Shew that when $|x| < 1$,

$$\int_c^{(x,\,0,\,x-,\,0-)} x^{1-\rho}(\nu-x)^{\rho-\alpha-1}\nu^{\alpha-1}(1-\nu)^{-\alpha_1}\,d\nu$$

$$= -4e^{\pi\rho i} \sin \alpha\pi \sin(\rho-\alpha)\pi \,.\, \frac{\Gamma(\rho-\alpha)\Gamma(\alpha)}{\Gamma(\rho)} F(\alpha, \alpha_1, \rho, x),$$

where c denotes a point on the finite line joining the points O, x, the initial arguments of $\nu-x$ and of ν are the same as that of x, and that of $(1-\nu)$ reduces to zero at the origin.

(Pochhammer.)

CHAPTER XII.

Bessel Functions.

146. *The Bessel coefficients.*

In this chapter we shall consider a class of functions known as *Bessel functions*, which present many analogies with the Legendre functions considered in Chapter X. As in the case of the Legendre functions, we shall first introduce the functions, or rather a certain set of them, as coefficients in an expansion.

For all finite values of z, and all finite values of t except $t=0$, the function

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$$

can be expanded by Laurent's theorem (§ 43) in a series of ascending and descending powers of t. If the coefficient of t^n, where n is any positive or negative integer, be denoted by $J_n(z)$, we have (by § 43)

$$J_n(z)=\frac{1}{2\pi i}\int u^{-n-1}e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)}\,du,$$

the integral being taken round any simple contour in the u-plane enclosing the point $u=0$.

To express this quantity $J_n(z)$ as a power-series in z, write

$$u=\frac{2t}{z}.$$

Thus
$$J_n(z)=\frac{1}{2\pi i}\left(\frac{z}{2}\right)^n\int t^{-n-1}e^{t-\frac{z^2}{4t}}\,dt,$$

the integral being taken round any simple contour in the t-plane enclosing the point $t=0$. This can be written

$$J_n(z)=\frac{1}{2\pi i}\left(\frac{z}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\left(\frac{z}{2}\right)^{2r}\int t^{-n-r-1}e^t\,dt.$$

Now (§ 56) we have

$\frac{1}{2\pi i}\int t^{-n-r-1}e^t dt=$ the residue of the function $t^{-n-r-1}e^t$ at its pole, the origin.

If n is a positive integer, this residue is

$$\frac{1}{(n+r)!};$$

if n is a negative integer, say $=-s$, the residue is zero when $r=0, 1, 2, \ldots s-1$, and when $r \geqslant s$ it is

$$\frac{1}{(r-s)!}.$$

In any case, the residue is

$$\frac{1}{\Gamma(n+r+1)}.$$

Thus if n is a positive integer, we have

$$J_n(z)=\sum_{r=0}^{\infty}\left(\frac{z}{2}\right)^{n+2r}\frac{(-1)^r}{r!\,(n+r)!};$$

and if n is a negative integer, equal to $-s$, we have

$$J_n(z)=\sum_{r=s}^{\infty}\left(\frac{z}{2}\right)^{2r-s}\frac{(-1)^r}{r!\,(r-s)!}=\sum_{t=0}^{\infty}\left(\frac{z}{2}\right)^{s+2t}\frac{(-1)^{s+t}}{(s+t)!\,t!},$$

or
$$J_n(z)=(-1)^s J_s(z).$$

Whether n be a positive or negative integer, the expansion can clearly be written in the form

$$J_n(z)=\sum_{r=0}^{\infty}\frac{(-1)^r z^{n+2r}}{2^{n+2r}r!\,\Gamma(n+r+1)}.$$

The function $J_n(z)$ thus defined for integral values of n is called the *Bessel coefficient* of the nth order.

We shall see subsequently (§ 149) that the Bessel coefficients are a particular case of a more extended class of functions known as *Bessel functions*.

Bessel coefficients were introduced by Bessel in 1824 in his "Untersuchung des Theils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht."

In reading some of the earlier papers on the subject, it is to be remembered that the notation has changed, what was formerly denoted by $J_n(z)$ being now denoted by $J_n(2z)$.

Example 1. Prove that if

$$\frac{2b(1+\theta^2)}{(1-2a\theta-\theta^2)^2+4b^2\theta^2}=A_1+A_2\theta+A_3\theta^2+\ldots,$$

then will
$$e^{az}\sin bz=A_1J_1(z)+A_2J_2(z)+A_3J_3(z)+\ldots.$$

(Cambridge Mathematical Tripos, Part I, 1896.)

For replacing the Bessel functions in the given series by their values as definite integrals, we have

$$A_1 J_1(z) + A_2 J_2(z) + A_3 J_3(z) + \ldots$$

$$= \frac{1}{2\pi i} \int e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} \left(\frac{A_1}{u^2} + \frac{A_2}{u^3} + \frac{A_3}{u^4} + \ldots\right) du$$

$$= \frac{1}{2\pi i} \int e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} \frac{2b\left(\frac{1}{u^2} + \frac{1}{u^4}\right)}{\left(1 - \frac{2a}{u} - \frac{1}{u^2}\right)^2 + \frac{4b^2}{u^2}} du,$$

the integrals being taken round any simple contour in the u-plane enclosing the origin.

Taking a new variable t, defined by the equation

$$t = \frac{1}{2}\left(u - \frac{1}{u}\right) - a,$$

we thus have

$$A_1 J_1(z) + A_2 J_2(z) + A_3 J_3(z) + \ldots = \frac{1}{2\pi i} \int \frac{e^{z(t+a)} b\,dt}{t^2 + b^2},$$

where the integration is now to be taken in the clockwise direction round any large simple contour in the t-plane. This expression is (§ 56) equal to minus the sum of the residues of the function

$$\frac{e^{z(t+a)} b}{t^2 + b^2}$$

at its poles $t = ib$ and $t = -ib$; that is, it is equal to

$$\frac{1}{2i} e^{z(a+ib)} - \frac{1}{2i} e^{z(a-ib)}$$

or

$$e^{az} \sin bz,$$

which is the required result.

Example 2. Shew that, when n is an integer,

$$J_n(z+y) = \sum_{m=-\infty}^{\infty} J_m(z) J_{n-m}(y).$$

We have

$$e^{\frac{1}{2}(z+y)\left(t - \frac{1}{t}\right)} = e^{\frac{z}{2}\left(t - \frac{1}{t}\right)} \cdot e^{\frac{y}{2}\left(t - \frac{1}{t}\right)},$$

or

$$\sum_{n=-\infty}^{\infty} t^n J_n(z+y) = \sum_{m=-\infty}^{\infty} t^m J_m(z) \sum_{r=-\infty}^{\infty} t^r J_r(y).$$

Equating coefficients of t^n on both sides of this equation, we have the required result.

147. *Bessel's differential equation.*

We have seen that, for all integer values of n, the Bessel coefficient of order n is expressed by the formula

$$J_n(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \int_C t^{-n-1} e^{t - \frac{z^2}{4t}} dt,$$

where C is a simple contour in the t-plane enclosing the point $t = 0$.

We shall now shew that the function $J_n(z)$ is a solution of a certain linear differential equation of the second order, namely,

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0.$$

For we find in performing the differentiations that

$$\begin{aligned}\frac{d^2J_n(z)}{dz^2}+\frac{1}{z}\frac{dJ_n(z)}{dz}+\left(1-\frac{n^2}{z^2}\right)J_n(z)\\ &=\frac{1}{2\pi i}\frac{z^n}{2^n}\int_C t^{-n-1}e^{t-\frac{z^2}{4t}}\left\{1-\frac{n+1}{t}+\frac{z^2}{4t^2}\right\}dt\\ &=-\frac{1}{2\pi i}\left(\frac{z}{2}\right)^n\int_C\frac{d}{dt}\left(e^{t-\frac{z^2}{4t}}t^{-n-1}\right)dt\\ &=0,\end{aligned}$$

since the function $e^{t-\frac{z^2}{4t}}t^{-n-1}$ resumes its original value after the point t has described the contour in question.

Thus $J_n(z)$ *satisfies the differential equation*

$$\frac{d^2J_n(z)}{dz^2}+\frac{1}{z}\frac{dJ_n(z)}{dz}+\left(1-\frac{n^2}{z^2}\right)J_n(z)=0.$$

This is called *Bessel's equation of order n*. Its properties in many respects resemble those of Legendre's differential equation, which is also a linear differential equation of the second order.

148. *Bessel's equation as a case of the hypergeometric equation.*

If c be any finite quantity, the differential equation of the hypergeometric function

$$P\begin{Bmatrix}0 & \infty & c & \\ n & ic & \frac{1}{2}+ic & z\\ -n & -ic & \frac{1}{2}-ic & \end{Bmatrix}$$

is (§ 137)

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(\frac{n^2}{z}+\frac{cz+\frac{1}{4}}{z-c}\right)\frac{cy}{z(z-c)}=0.$$

If in this equation we make c tend to an infinitely large value, we obtain

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0,$$

which is Bessel's equation of order n. Thus *Bessel's equation can be regarded as a limiting case of the hypergeometric equation*, corresponding to the function

$$\operatorname*{Limit}_{c=\infty} P \begin{Bmatrix} 0 & \infty & c & \\ n & ic & \frac{1}{2}+ic & z \\ -n & -ic & \frac{1}{2}-ic & \end{Bmatrix}.$$

Another representation of Bessel's equation as a limiting case of the hypergeometric equation is the following.

If we change the dependent variable in Bessel's equation, by writing $y = e^{iz}u$, the differential equation for u is easily found to be

$$\frac{d^2u}{dz^2} + \left(2i + \frac{1}{z}\right)\frac{du}{dz} + \left(\frac{i}{z} - \frac{n^2}{z^2}\right)u = 0.$$

Now if c be any quantity, the differential equation of the hypergeometric function

$$P \begin{Bmatrix} 0 & \infty & c & \\ n & \frac{1}{2} & 0 & z \\ -n & \frac{3}{4} - 2ic & 2ic - 1 & \end{Bmatrix}$$

is
$$\frac{d^2u}{dz^2} + \left(\frac{1}{z} + \frac{2-2ic}{z-c}\right)\frac{du}{dz} + \left(\frac{n^2c}{z} + \frac{3}{8} - ic\right)\frac{u}{z(z-c)} = 0.$$

If in this equation we make c tend to infinity, we obtain

$$\frac{d^2u}{dz^2} + \left(\frac{1}{z} + 2i\right)\frac{du}{dz} + \left(-\frac{n^2}{z^2} + \frac{i}{z}\right)u = 0,$$

which is the above equation. Hence Bessel's equation is a limiting case of the hypergeometric equation, being the equation for the function

$$e^{iz} \operatorname*{Limit}_{c=\infty} P \begin{Bmatrix} 0 & \infty & c & \\ n & \frac{1}{2} & 0 & z \\ -n & \frac{3}{4} - 2ic & 2ic - 1 & \end{Bmatrix}.$$

Bessel's equation is connected not merely with the general hypergeometric equation, but with that special form of it which we have considered in connexion with the Legendre functions.

For the differential equation of the associated Legendre function (§ 129)

$$P_n^m\left(1-\frac{z^2}{2n^2}\right)$$

is (§ 138) the equation of the function

$$P\left\{\begin{matrix} -1 & \infty & 1 & \\ \frac{m}{2} & n+1 & \frac{m}{2} & 1-\frac{z^2}{2n^2} \\ -\frac{m}{2} & -n & -\frac{m}{2} & \end{matrix}\right\},$$

or (§ 139)

$$P\left\{\begin{matrix} 4n^2 & \infty & 0 & \\ \frac{m}{2} & n+1 & \frac{m}{2} & z^2 \\ -\frac{m}{2} & -n & -\frac{m}{2} & \end{matrix}\right\}.$$

The differential equation of this function is

$$\frac{d^2y}{d(z^2)^2}+\left(\frac{1}{z^2-4n^2}+\frac{1}{z^2}\right)\frac{dy}{d(z^2)}+\left(-\frac{m^2}{z^2-4n^2}-\frac{n+1}{n}+\frac{m^2}{z^2}\right)\frac{n^2y}{z^2(z^2-4n^2)}=0.$$

If in this equation we make n tend to infinity, it becomes

$$\frac{d^2y}{d(z^2)^2}+\frac{1}{z^2}\frac{dy}{d(z^2)}-\left(-1+\frac{m^2}{z^2}\right)\frac{y}{4z^2}=0,$$

or

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{m^2}{z^2}\right)y=0,$$

which is Bessel's equation. Thus Bessel's equation of order m is the same as the equation for the function

$$\operatorname*{Limit}_{n=\infty} P_n^m\left(1-\frac{z^2}{2n^2}\right).$$

By considering Bessel's equation as a limiting case of the hypergeometric equation, we can deduce certain solutions in the form of definite integrals.

For the differential equation of the function

$$P\left\{\begin{matrix} 0 & \infty & c & \\ n & -ic & \frac{1}{2}+ic & z \\ -n & +ic & \frac{1}{2}-ic & \end{matrix}\right\}$$

is satisfied by the integral

$$z^n\left(1-\frac{z}{c}\right)^{\frac{1}{2}+ic}\int_C t^{-n-\frac{1}{2}}\left(1-\frac{t}{c}\right)^{n+\frac{1}{2}-2ic}(t-z)^{-n-\frac{1}{2}}\,dt,$$

if C is a contour such that after describing C the integrand returns to its initial value. When c becomes infinite, this expression reduces to

$$z^n e^{-zi} \int_C t^{-n-\frac{1}{2}} (t-z)^{-n-\frac{1}{2}} e^{2it}\, dt,$$

which accordingly satisfies Bessel's equation if C be a contour of the kind described; C can for instance be a figure-of-eight contour encircling the points $t=0$ and $t=z$.

In fact, if we write

$$y = z^n e^{-zi} \int_C t^{-n-\frac{1}{2}} (t-z)^{-n-\frac{1}{2}} e^{2it}\, dt,$$

we have

$$\begin{aligned} &\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right) y \\ &\qquad = -\left(n + \frac{1}{2}\right) z^{n-1} e^{iz} \int \frac{d}{dt} \{t^{\frac{1}{2}-n} e^{2it} (t-z)^{-n-\frac{3}{2}}\}\, dt \\ &\qquad = 0. \end{aligned}$$

Other solutions can be found by changing the signs of n and i.

Example. Shew that Bessel's differential equation is the limiting case of the equation of the hypergeometric function

$$P\left\{\begin{matrix} 0 & \infty & c^2 & \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 & z^2 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 & \end{matrix}\right\}$$

when c tends to infinity.

149. *The general solution of Bessel's equation by Bessel functions whose order is not necessarily an integer.*

We now proceed, in the same way as in § 116, to extend our definition of the function $J_n(z)$ to the general case in which n is not an integer.

It appears from the proof given in § 147 that, whatever n may be, the differential equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right) y = 0$$

is satisfied by an integral of the form

$$y = z^n \int_C t^{-n-1} e^{t - \frac{z^2}{4t}}\, dt,$$

provided the path of integration C is a contour on the t-plane, so chosen that the function

$$e^{t-\frac{z^2}{4t}} t^{-n-1}$$

resumes its initial value after describing C.

Now when the real part of t is a very large negative number, the function

$$e^{t-\frac{z^2}{4t}}t^{-n-1}$$

is infinitesimal. Hence y will be a solution of the differential equation, provided the contour C begins and ends with values of t whose real part is infinitely large and negative.

Let therefore a contour C be taken which begins at the negative end of the real axis, and after proceeding close to the real axis to the neighbourhood of the origin makes a circuit of the origin and returns, close to the real axis, to the negative end of the real axis again. *The integral y taken round this contour satisfies Bessel's differential equation.*

We shall now shew that this solution y can be expressed in the form of a series of powers of z.

Suppose as usual that by t^{-n-1} is understood that branch of the function t^{-n-1} which when continued (§ 41) to the point $t=1$ by a straight path, arrives at the point $t=1$ with the value unity.

Then we have

$$y = z^n \int_C t^{-n-1} e^t \,.\, e^{-\frac{z^2}{4t}}\, dt$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+n}}{2^{2r}\,.\,r!} \int_C e^t t^{-n-1-r}\, dt.$$

But (§ 100) we have

$$\frac{1}{\Gamma(k)} = \frac{1}{2\pi i} \int_C t^{-k} e^t\, dt.$$

Thus

$$y = \sum_{r=0}^{\infty} \frac{2\pi i\,.\,(-1)^r\,.\,z^{2r+n}}{2^{2r}.r!\,\Gamma(r+n+1)}.$$

But when n is an integer, we have (§ 146)

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+n}}{2^{n+2r} r!\,\Gamma(n+r+1)}.$$

Comparing these results, we have, when n is an integer,

$$J_n(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \int_C t^{-n-1} e^{t-\frac{z^2}{4t}}\, dt,$$

where C is the contour already described.

Now we have seen that the right-hand side of this equation has a meaning and satisfies Bessel's differential equation for all values of z and all values of n; whereas, up to the present, $J_n(z)$ has been defined only for integral values

of n. We shall take this opportunity of extending the definition of $J_n(z)$, in the following way.

For all values of n and of z, the function

$$\frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \int_C t^{-n-1} e^{t-\frac{z^2}{4t}}\, dt$$

or

$$\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+n}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

will be denoted by $J_n(z)$. In the integral, t^{-n-1} is to have the value which becomes unity when the variable t travels in a straight line to the point $t = 1$: and C is a contour which encircles the point $t = 0$ and begins and ends at the negative end of the real axis in the t-plane. The function $J_n(z)$ thus defined is called the *Bessel function* of z, of the first kind and of the order n; *it satisfies Bessel's differential equation of order n.*

Since Bessel's differential equation is unaltered by the change of n into $-n$, we see that $J_{-n}(z)$ is also a solution of the equation; and therefore *the general solution of Bessel's equation is of the form*

$$aJ_n(z) + bJ_{-n}(z),$$

where a and b are arbitrary constants, except in the case in which $J_n(z)$ and $J_{-n}(z)$ are not independent functions; this exceptional case happens when n is an integer, for then, as we have already seen, we have the relation

$$J_n(z) = (-1)^n J_{-n}(z).$$

A second solution of Bessel's equation in the case when n is an integer will be given later.

150. *The recurrence-formulae for the Bessel functions.*

As the Bessel functions, like the Legendre functions, are members of the general class of hypergeometric functions, it is to be expected that recurrence-formulae will exist between them, corresponding to the relations between contiguous hypergeometric functions (§ 145).

We shall now establish these recurrence-relations; the proof given does not assume the order n to be an integer, and consequently the formulae are valid for all values of n, real or complex.

Let C be the contour described in the last article, which begins and ends at the negative end of the real axis in the t-plane, and encircles the point $t = 0$.

Then since the function

$$e^{t-\frac{z^2}{4t}} t^{-n}$$

is infinitesimal at the extremities of this contour, we have the equation

$$0 = \int_C \frac{d}{dt}\left(t^{-n} e^{t-\frac{z^2}{4t}}\right) dt$$

$$= \int_C e^{t-\frac{z^2}{4t}}\left(t^{-n} + \frac{1}{4}z^2 t^{-n-2} - nt^{-n-1}\right) dt$$

$$= 2\pi i \left\{\left(\frac{2}{z}\right)^{n-1} J_{n-1}(z) + \frac{1}{4}z^2 \left(\frac{2}{z}\right)^{n+1} J_{n+1}(z) - n\left(\frac{2}{z}\right)^n J_n(z)\right\}$$

or

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z) \quad \text{.........................(A)},$$

which is the first of the recurrence-formulae.

Next we have, by differentiation,

$$\frac{d}{dz}\int_C t^{-n-1} e^{t-\frac{z^2}{4t}} dt = -\frac{1}{2}z \int_C t^{-n-2} e^{t-\frac{z^2}{4t}} dt,$$

or

$$\frac{d}{dz}\left\{\left(\frac{2}{z}\right)^n J_n(z)\right\} = -\left(\frac{2}{z}\right)^n J_{n+1}(z),$$

or

$$\frac{dJ_n(z)}{dz} = \frac{n}{z} J_n(z) - J_{n+1}(z) \quad \text{.............................(B)}.$$

From (A) and (B) it is easy to derive other recurrence-formulae, e.g.

$$\frac{dJ_n(z)}{dz} = \frac{1}{2}\{J_{n-1}(z) - J_{n+1}(z)\} \quad \text{.........................(C)},$$

and

$$\frac{dJ_n(z)}{dz} = J_{n-1}(z) - \frac{n}{z} J_n(z) \quad \text{..............................(D)}.$$

Example 1. Shew that

$$16\frac{d^4 J_n(z)}{dz^4} = J_{n-4}(z) - 4J_{n-2}(z) + 6J_n(z) - 4J_{n+2}(z) + J_{n+4}(z).$$

Example 2. Shew that

$$J_2(z) = -\frac{dJ_0(z)}{dz} + \frac{d^2 J_0(z)}{dz^2}.$$

151. *Relation between two Bessel functions whose orders differ by an integer.*

The various recurrence-formulae found in the last article can however be easily deduced from a single equation, which connects any two Bessel functions whose orders differ by an integer, namely

$$\frac{J_{n+r}(z)}{z^{n+r}} = (-1)^r \frac{d^r}{(z\,dz)^r}\left\{\frac{J_n(z)}{z^n}\right\},$$

where n is any number (real or complex) and r is any positive integer.

To establish this result, we have, by § 149,

$$\frac{d^r}{d(z^2)^r}\left\{\frac{J_n(z)}{z^n}\right\} = \frac{d^r}{d(z^2)^r}\frac{1}{2\pi i \,.\, 2^n}\int_C t^{-n-1} e^{t-\frac{z^2}{4t}}\,dt$$

$$= \frac{1}{2\pi i \,.\, 2^n}\int_C \frac{t^{-n-1}}{(-4t)^r} e^{t-\frac{z^2}{4t}}\,dt$$

$$= \frac{1}{(-2)^r}\frac{1}{2\pi i \,.\, 2^{n+r}}\int_C t^{-n-r-1} e^{t-\frac{z^2}{4t}}\,dt$$

$$= \frac{1}{(-2)^r}\frac{1}{z^{n+r}} J_{n+r}(z),$$

which is the equation required.

The recurrence-formulae can be derived without difficulty from this result. Thus, equation (B) of the last article is obtained by taking $r = 1$ in this equation: and equation (A) of the last article may be derived in the following way.

Taking $r = 1$ and $r = 2$ successively in the formula just proved, we can express the first and second derivates of $J_n(z)$ in terms of $J_n(z)$, $J_{n+1}(z)$ and $J_{n+2}(z)$, in the form

$$\frac{dJ_n(z)}{dz} = \frac{n}{z} J_n(z) - J_{n+1}(z),$$

$$\frac{d^2J_n(z)}{dz^2} = -\frac{n}{z^2}(1-n) J_n(z) - \frac{2n+1}{z} J_{n+1}(z) + J_{n+2}(z).$$

Substituting these values in Bessel's equation

$$\frac{d^2J_n(z)}{dz^2} + \frac{1}{z}\frac{dJ_n(z)}{dz} + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0,$$

we have
$$J_{n+2}(z) - \frac{2(n+1)}{z} J_{n+1}(z) + J_n(z) = 0.$$

Changing n to $(n-1)$ in this result, we have

$$J_{n+1}(z) - \frac{2n}{z} J_n(z) + J_{n-1}(z) = 0,$$

which is the formula (A) of the last article.

The other recurrence-formulae can be derived in a similar way.

152. *The roots of Bessel functions.*

The relation established in the preceding article enables us to deduce the interesting theorem that *between any two consecutive real roots of $J_n(z)$ there lies one and only one root of $J_{n+1}(z)$**.

For since $J_n(z)$ satisfies Bessel's equation, it follows that the function $y = z^{-n} J_n(z)$ satisfies the differential equation

$$\frac{d^2}{dz^2}(z^n y) + \frac{1}{z}\frac{d}{dz}(z^n y) + \left(1 - \frac{n^2}{z^2}\right) z^n y = 0$$

or

$$z\frac{d^2y}{dz^2} + (2n+1)\frac{dy}{dz} + zy = 0.$$

From this equation it is evident that if ξ be a value of z (real and not zero) for which $\frac{dy}{dz}$ is zero, then the signs of $\frac{d^2y}{dz^2}$ and y must be unlike at the point $z = \xi$. Now let $z = \xi_1$ and $z = \xi_2$ be two consecutive roots of the function $\frac{dy}{dz}$. It is clear from the differential equation that neither y nor $\frac{d^2y}{dz^2}$ can be zero at either of these points. Then the function $\frac{dy}{dz}\frac{d^2y}{dz^2}$ has a different sign just before reaching $z = \xi_2$ to that which it has just after leaving $z = \xi_1$; and hence it follows that the function $y\frac{dy}{dz}$ has a different sign just before reaching $z = \xi_2$ to that which it has just after leaving $z = \xi_1$. The function y must therefore have an odd number of roots between the points $z = \xi_1$ and $z = \xi_2$.

But from Rolle's Theorem it follows that y cannot be zero more than once in this interval: so y must have one and only one zero between the points $z = \xi_1$ and $z = \xi_2$: and therefore the zeros of y and of $\frac{dy}{dz}$ occur alternately.

Thus, between any two consecutive roots of the function $z^{-n} J_n(z)$ there lies one and only one root of the function $\frac{d}{dz}\{z^{-n} J_n(z)\}$ or $-z^{-n} J_{n+1}(z)$: which establishes the theorem.

153. *Expression of the Bessel coefficients as trigonometric integrals.*

We shall next obtain a form for the Bessel coefficients (i.e. the Bessel functions for which the order n is an integer), which in some respects corresponds to the Laplacian integrals obtained in §§ 119 and 132 for the Legendre functions.

* The proof here given is due to Gegenbauer, *Monatshefte für Math.* VIII. (1897).

If in the equation

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$$

we write $t = e^{i\phi}$, we have

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{ni\phi}.$$

Changing i to $-i$ in this equation, we have

$$e^{-iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{-ni\phi}.$$

Adding and subtracting these results, we have

$$\cos(z\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(z)\cos n\phi,$$

$$\sin(z\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(z)\sin n\phi.$$

Since $J_n(z) = (-1)^n J_{-n}(z)$, these equations give

$$\cos(z\sin\phi) = J_0(z) + 2J_2(z)\cos 2\phi + 2J_4(z)\cos 4\phi + \ldots,$$

$$\sin(z\sin\phi) = 2J_1(z)\sin\phi + 2J_3(z)\sin 3\phi + \ldots.$$

As these are Fourier series, we have (§ 82)

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos n\theta\cos(z\sin\theta)\, d\theta, \quad (n \text{ even}),$$

$$0 = \frac{1}{\pi}\int_0^\pi \cos n\theta\cos(z\sin\theta)\, d\theta, \quad (n \text{ odd}),$$

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \sin n\theta\sin(z\sin\theta)\, d\theta, \quad (n \text{ odd}),$$

$$0 = \frac{1}{\pi}\int_0^\pi \sin n\theta\sin(z\sin\theta)\, d\theta, \quad (n \text{ even}).$$

Since

$$\cos(n\theta - z\sin\theta) = \cos n\theta\cos(z\sin\theta) + \sin n\theta\sin(z\sin\theta),$$

we have in all cases when n is an integer

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta,$$

the formula required.

Example. To shew that for all values of n, real or complex, the integral

$$y = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta$$

satisfies the differential equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=\frac{\sin n\pi}{\pi}\left(\frac{1}{z}-\frac{n}{z^2}\right),$$

which reduces to Bessel's equation when n is an integer.

For if
$$y=\frac{1}{\pi}\int_0^\pi \cos(n\theta-z\sin\theta)\,d\theta,$$

we have
$$\frac{dy}{dz}=\frac{1}{\pi}\int_0^\pi \sin\theta\sin(n\theta-z\sin\theta)\,d\theta,$$

$$\frac{d^2y}{dz^2}=-\frac{1}{\pi}\int_0^\pi \sin^2\theta\cos(n\theta-z\sin\theta)\,d\theta,$$

so
$$y+\frac{d^2y}{dz^2}=\frac{1}{\pi}\int_0^\pi \cos^2\theta\cos(n\theta-z\sin\theta)\,d\theta,$$

and
$$\frac{1}{z}\frac{dy}{dz}-\frac{n^2}{z^2}y=\frac{1}{\pi}\int_0^\pi \frac{\sin\theta}{z}\sin(n\theta-z\sin\theta)\,d\theta-\frac{1}{\pi}\int_0^\pi \frac{n^2}{z^2}\cos(n\theta-z\sin\theta)\,d\theta.$$

Now integrating by parts, we have

$$\frac{1}{\pi}\int_0^\pi \frac{\sin\theta}{z}\sin(n\theta-z\sin\theta)\,d\theta=\frac{1}{\pi z}\sin n\pi+\frac{1}{\pi}\int_0^\pi \frac{\cos\theta}{z}(n-z\cos\theta)\cos(n\theta-z\sin\theta)\,d\theta,$$

and therefore

$$\begin{aligned}\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y&=\frac{1}{\pi z}\sin n\pi+\frac{1}{z^2\pi}\int_0^\pi (nz\cos\theta-n^2)\cos(n\theta-z\sin\theta)\,d\theta\\&=\frac{1}{\pi z}\sin n\pi-\frac{n}{z^2\pi}\int_{\theta=0}^{\theta=\pi}\cos(n\theta-z\sin\theta)\,.\,d(n\theta-z\sin\theta)\\&=\frac{1}{\pi z}\sin n\pi-\frac{n}{z^2\pi}\sin n\pi\\&=\frac{\sin n\pi}{\pi}\left(\frac{1}{z}-\frac{n}{z^2}\right),\end{aligned}$$

which is the required result.

154. *Extension of the integral-formula to the case in which n is not an integer.*

We shall now shew how the result

$$J_n(z)=\frac{1}{\pi}\int_0^\pi \cos(n\theta-z\sin\theta)\,d\theta$$

must be generalised in order to meet the case in which n is not an integer, i.e. the case of the Bessel *functions*, as opposed to the Bessel *coefficients*.

Suppose that the real part of z is positive. Write $t = \frac{1}{2} zu$ in the formula

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \int_C t^{-n-1} e^{t - \frac{z^2}{4t}} dt;$$

we thus have

$$J_n(z) = \frac{1}{2\pi i}\int u^{-n-1} e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} du,$$

where u^{-n-1} has that value which becomes unity when the variable u travels by a rectilinear path to the point $u = 1$. Since values of t whose real part is large and negative correspond to values of u whose real part is large and negative, we see that the path in the u-plane, along which this integral is to be taken, is still a path leading from $u = -\infty$ round the point $u = 0$ and returning to $u = -\infty$.

Let this contour be chosen so as to consist of

(α) a straight line parallel to, and below, but indefinitely close to, the real axis from $u = -\infty$ to $u = -1$;

(β) a circle l of radius unity described round the origin;

(γ) a straight line parallel to, and above, but indefinitely close to, the real axis from $u = -1$ to $u = -\infty$.

Thus

$$J_n(z) = \frac{1}{2\pi i}\int_{-\infty}^{-1} u^{-n-1} e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} du + \frac{1}{2\pi i}\int_l u^{-n-1} e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} du$$
$$+ \frac{1}{2\pi i}\int_{-1}^{-\infty} u^{-n-1} e^{\frac{z}{2}\left(u - \frac{1}{u}\right)} du,$$

where u^{-n-1} has in the first integral the value $e^{(n+1)i\pi}$ at $u = -1$, and in the third integral has the value $e^{-(n+1)i\pi}$ at $u = -1$. Hence, writing $u = -t$ in the first and third integrals, and $u = e^{i\theta}$ in the second integral, we have

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta + zi\sin\theta} d\theta + \frac{e^{(n+1)i\pi}}{2\pi i}\int_1^{\infty} t^{-n-1} e^{\frac{z}{2}\left(-t + \frac{1}{t}\right)} dt$$
$$- \frac{e^{-(n+1)i\pi}}{2\pi i}\int_1^{\infty} t^{-n-1} e^{\frac{z}{2}\left(-t + \frac{1}{t}\right)} dt,$$

where, in the last two integrals, t^{-n-1} has the value 1 at the point $t = 1$. Writing $t = e^{\theta}$, we have

$$J_n(z) = \frac{1}{2\pi}\left\{\int_0^{\pi} e^{-ni\theta + zi\sin\theta} d\theta + \int_{-\pi}^{0} e^{-ni\theta + zi\sin\theta} d\theta\right\}$$
$$+ \frac{\sin(n+1)\pi}{\pi}\int_0^{\infty} e^{-n\theta - z\sinh\theta} d\theta$$
$$= \frac{1}{2\pi}\int_0^{\pi}\left\{e^{i(z\sin\theta - n\theta)} + e^{-i(z\sin\theta - n\theta)}\right\} d\theta + \frac{\sin(n+1)\pi}{\pi}\int_0^{\infty} e^{-n\theta - z\sinh\theta} d\theta,$$

or $$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\theta - z\sinh\theta}\,d\theta \quad\text{.........(1).}$$

This formula is valid when the real part of z is positive. When the real part of z is negative, a similar procedure leads to the result

$$J_n(z) = \frac{e^{n\pi i}}{\pi}\left\{\int_0^\pi \cos(z\sin\theta + n\theta)\,d\theta - \sin n\pi\int_0^\infty e^{-n\theta + z\sinh\theta}\,d\theta\right\} \quad\text{......(2).}$$

When n is an integer, the formula (1) gives

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta,$$

when the real part of z is positive; and the formula (2) gives

$$J_n(z) = \frac{(-1)^n}{\pi}\int_0^\pi \cos(n\theta + z\sin\theta)\,d\theta,$$

or, since $$J_n(z) = (-1)^n J_{-n}(z),$$

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta,$$

when the real part of z is negative.

Thus in either case when n is an integer, we have again the result of the last article, namely the formula

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta \quad\text{.....................(3).}$$

The equation (3) was known to Bessel. Equation (1) is due to Schläfli, *Math. Ann.* III. (1871); equation (2) was first given by Sonine, *Math. Ann.* XVI. (1880).

The trigonometric integral-formula for $J_n(z)$ may be regarded as corresponding to the Laplacian definite integrals for the Legendre functions. For we have seen that the Bessel function $J_m(z)$ satisfies the differential equation of the function

$$\operatorname*{Limit}_{n=\infty} P_n^m\left(1 - \frac{z^2}{2n^2}\right).$$

But the Laplacian integral shews that this quantity is a multiple of

$$\operatorname*{Limit}_{n=\infty}\int_0^\pi \left[1 - \frac{z^2}{2n^2} + \left\{\left(1 - \frac{z^2}{2n^2}\right)^2 - 1\right\}^{\frac{1}{2}}\cos\phi\right]^n \cos m\phi\,d\phi$$

or $$\operatorname*{Limit}_{n=\infty}\int_0^\pi \left(1 + \frac{iz}{n}\cos\phi\right)^n \cos m\phi\,d\phi,$$

or $$\int_0^\pi e^{iz\cos\phi}\cos m\phi\,d\phi,$$

the similarity of which to the above result (3) will be observed.

155. *A second expression of $J_n(z)$ as a definite integral whose path of integration is real.*

Another definite-integral formula, which is valid for all values of z and a certain range of values of n, can be obtained in the following way.

The function $J_n(z)$ is expressed for all values of n and z by the series

$$\sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}.$$

Since (§ 95) we have

$$\Gamma\left(r+\frac{1}{2}\right) = \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2r!}{2^{2r}\,.\,r!}\,\Gamma\left(\frac{1}{2}\right),$$

this can be written in the form

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}\, \Gamma\left(r+\frac{1}{2}\right)}{2^n\, \Gamma(n+r+1)\,.\,2r!\,\Gamma\left(\frac{1}{2}\right)}.$$

Now by § 107 we have

$$\int_0^\pi \cos^{2r}\phi \sin^{2n}\phi\, d\phi = \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(r+n+1)},$$

provided the real parts of $\left(r+\frac{1}{2}\right)$ and $\left(n+\frac{1}{2}\right)$ are positive.

Thus if the real part of $\left(n+\frac{1}{2}\right)$ be positive, we have

$$J_n(z) = \frac{1}{2^n\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2r!} \int_0^\pi \cos^{2r}\phi \sin^{2n}\phi\, d\phi.$$

But $$\cos(z\cos\phi) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}\cos^{2r}\phi}{2r!}.$$

Thus we have

$$J_n(z) = \frac{z^n}{2^n\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(z\cos\phi)\sin^{2n}\phi\, d\phi.$$

This formula is true for all values of z, and for all values of n whose real part is greater than $-\frac{1}{2}$.

Example 1. Shew that

$$P_n(\cos\theta) = \frac{1}{\Gamma(n+1)} \int_0^\infty e^{-x\cos\theta} J_0(x\sin\theta)\, x^n\, dx.$$

For we have

$$J_0(x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\phi}\,d\phi,$$

so

$$\int_0^\infty e^{-x\cos\theta} J_0(x\sin\theta)\, x^n\,dx = \frac{1}{\pi}\int_0^\pi d\phi \int_0^\infty e^{-x\cos\theta}\,.\,e^{-ix\sin\theta\cos\phi}\,x^n\,dx$$

$$= \frac{1}{\pi}\int_0^\pi d\phi\, \frac{\Gamma(n+1)}{(\cos\theta + i\sin\theta\cos\phi)^{n+1}}$$

$$= \Gamma(n+1)\,P_n(\cos\theta),$$

which establishes the result.

Example 2. Shew that

$$P_n^m(\cos\theta) = \frac{1}{\Gamma(n-m+1)}\int_0^\infty e^{-x\cos\theta} J_m(x\sin\theta)\,x^n\,dx.$$

(Cambridge Mathematical Tripos, Part II, 1893.)

156. *Hankel's definite-integral solution of Bessel's differential equation.*

If in the result of the last article we write

$$t = \cos\phi,$$

we obtain the result

$$J_n(z) = \frac{z^n}{2^n\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_{-1}^1 \cos(zt)\,(1-t^2)^{n-\frac{1}{2}}\,dt.$$

It will now be shewn that this integral is a member of a very general class of definite integrals which satisfy Bessel's differential equation, namely, integrals of the form

$$y = z^n \int_C e^{zti}\,(t^2-1)^{n-\frac{1}{2}}\,dt,$$

where C may be any one of a number of contours in the t-plane. The importance of solutions of this type was first shewn by Hankel*.

To shew that integrals of this class satisfy Bessel's equation, we form the first and second derivates of the expression y, and find that

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y$$

$$= -z^{n-1}\int_C \{z e^{zti}(t^2-1)^{n+\frac{1}{2}} - (2n+1)\,ti\,e^{zti}\,(t^2-1)^{n-\frac{1}{2}}\}\,dt$$

$$= iz^{n-1}\int_C \frac{d}{dt}\{e^{zti}(t^2-1)^{n+\frac{1}{2}}\}\,dt.$$

* *Math. Ann.* I.

From this it is clear that Bessel's equation will be satisfied by the integral

$$y = z^n \int_C e^{zti} (t^2 - 1)^{n-\frac{1}{2}} dt,$$

provided C is a closed contour such that the integrand resumes its initial value after making a circuit of C.

The similarity of this result to the general theorem of § 142 is very apparent.

157. *Expression of $J_n(z)$, for all values of n and z, by an integral of Hankel's type.*

We shall now shew how the particular solution $J_n(z)$ of Bessel's equation can be expressed by an integral of Hankel's type. Consider the contour formed by a figure-of-eight in the t-plane, enclosing the point $t = +1$ in one loop and the point $t = -1$ in the other, so that a description of the contour in the positive sense involves a turn in the positive direction round the point $t = +1$ and a turn in the negative direction round the point $t = -1$. After turning round the point $t = +1$ in the positive sense, the integrand resumes its original value multiplied by $e^{(n-\frac{1}{2})2\pi i}$, as can be seen by writing it in the form

$$e^{zti + (n-\frac{1}{2})\log(t-1) + (n-\frac{1}{2})\log(t+1)};$$

and after turning round $t = -1$ in the negative sense, it is further multiplied by

$$e^{-(n-\frac{1}{2})2\pi i}.$$

Hence after describing the whole contour, the integrand resumes its original value.

Thus
$$y = z^n \int^{(1+,\,-1-)} e^{zti} (t^2 - 1)^{n-\frac{1}{2}} dt$$

is a solution of the differential equation, valid for all values of z and of n; the symbol $(1+, -1-)$ placed at the upper limit of the integral indicating that the path of integration consists of a positive revolution round 1 and a negative revolution round -1

In this equation we shall suppose as usual that z^n has the value which reduces to 1 when z travels by a straight path to the point $z = 1$, and we shall suppose $(t^2 - 1)^{n-\frac{1}{2}}$ to have initially the value which reduces to $e^{-(n-\frac{1}{2})\pi i}$ when t travels by a straight path to the point $t = 0$.

To find the relation between this quantity y and the particular solution $J_n(z)$ of Bessel's equation, we expand y in the form

$$y = \sum_{r=0}^{\infty} \frac{z^{n+r} i^r}{r!} \int^{(1+,\,-1-)} t^r (t^2 - 1)^{n-\frac{1}{2}} dt.$$

To evaluate the integrals which occur in this series, write

$$F(r, n) = \int^{(1+, -1-)} t^r (t^2 - 1)^{n-\frac{1}{2}} dt.$$

Then $F(r, n+1) = \int^{(1+, -1-)} (t^{r+2} - t^r)(t^2 - 1)^{n-\frac{1}{2}} dt$

$$= \int^{(1+, -1-)} \frac{t^{r+1}}{2n+1} d\{(t^2 - 1)^{n+\frac{1}{2}}\} - F(r, n)$$

$$= -\int^{(1+, -1-)} \frac{(t^2 - 1)^{n+\frac{1}{2}}(r+1)}{2n+1} t^r dt - F(r, n)$$

$$= -\frac{(r+1)}{2n+1} F(r, n+1) - F(r, n).$$

Thus we have $\quad F(r, n) = -\dfrac{2n+r+2}{2n+1} F(r, n+1).$

This result enables us to reduce the evaluation of $F(r, n)$ to the evaluation of $F(r, n+1)$, and thus to the evaluation of $F(r, n+k)$, where k is a positive integer so chosen that the real part of $(n+k)$ is greater than $-\frac{1}{2}$.

We have therefore to evaluate the integral

$$F(r, n) = \int^{(1+, -1-)} t^r (t^2 - 1)^{n-\frac{1}{2}} dt,$$

where we may now suppose that the real part of n is greater than $-\frac{1}{2}$. The contour can be supposed to start at the point $t = 0$, where $(t^2 - 1)^{n-\frac{1}{2}}$ has the value $e^{-(n-\frac{1}{2})\pi i}$, then to proceed to the neighbourhood of the point $t = 1$ along the real axis, then to make a positive turn in a small circle round $t = 1$, then to return along the real axis to the point $t = 0$, where $(t^2 - 1)^{n-\frac{1}{2}}$ has now the value $e^{(n-\frac{1}{2})\pi i}$, then to proceed along the real axis to the neighbourhood of the point $t = -1$, then to make a negative turn in a small circle round $t = -1$, and lastly to return along the real axis to the point $t = 0$, where $(t^2 - 1)^{n-\frac{1}{2}}$ has now the value $e^{-(n-\frac{1}{2})\pi i}$. Since the real part of n is greater than $-\frac{1}{2}$, the integrals round the small circles at $t = 1$ and $t = -1$ are infinitesimal, and we therefore have

$$F(r, n) = e^{-(n-\frac{1}{2})\pi i} \int_0^1 t^r (1 - t^2)^{n-\frac{1}{2}} dt - e^{(n-\frac{1}{2})\pi i} \int_0^1 t^r (1 - t^2)^{n-\frac{1}{2}} dt$$

$$+ e^{(n-\frac{1}{2})\pi i} \int_0^{-1} t^r (1 - t^2)^{n-\frac{1}{2}} dt - e^{-(n-\frac{1}{2})\pi i} \int_0^{-1} t^r (1 - t^2)^{n-\frac{1}{2}} dt,$$

where in each of these integrals the quantity $(1 - t^2)^{n-\frac{1}{2}}$ is now supposed to

have the value unity at $t=0$. Writing $-t$ for t in the two last integrals, we have

$$F(r,\ n) = -\left\{e^{(n-\frac{1}{2})\pi i} - e^{-(n-\frac{1}{2})\pi i}\right\}\{1-(-1)^{r+1}\}\int^1 t^r (1-t^2)^{n-\frac{1}{2}}\,dt$$

$$= -2i\cos^2\frac{r\pi}{2}\sin\left(n-\frac{1}{2}\right)\pi\int_0^1 \nu^{\frac{1}{2}(r-1)}(1-\nu)^{n-\frac{1}{2}}\,d\nu, \text{ where } \nu = t^2,$$

(by § 105) $$= -2i\cos^2\frac{r\pi}{2}\sin\left(n-\frac{1}{2}\right)\pi B\left(\frac{r+1}{2},\ n+\frac{1}{2}\right)$$

(by § 106) $$= -2i\cos^2\frac{r\pi}{2}\sin\left(n-\frac{1}{2}\right)\pi\,\frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{r}{2}+1\right)}.$$

This result has now been proved to hold so long as the real part of n is greater than $-\frac{1}{2}$: and in virtue of the formula

$$F(r,\ n) = -\frac{2n+r+2}{2n+1}F(r,\ n+1),$$

we see that it holds universally.

Thus we have $F(r, n) = 0$, when r is odd; and it is therefore sufficient to take r even. Let $r = 2s$. Then the formula becomes

$$F(2s,\ n) = 2i\sin\left(n+\frac{1}{2}\right)\pi\,\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+s+1)}.$$

But (§ 97) we have

$$\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-n\right) = \frac{\pi}{\sin\left(n+\frac{1}{2}\right)\pi}.$$

Therefore $$F(2s,\ n) = \frac{2i\pi\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(n+s+1)},$$

and so $$y = \sum_{s=0}^{\infty}\frac{(-1)^s z^{n+2s}}{2s!}\,\frac{2i\pi\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(n+s+1)},$$

or $$y = \sum_{s=0}^{\infty}\frac{(-1)^s z^{n+2s}}{2^{2s}s!}\,\frac{2i\pi\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(n+s+1)}.$$

But $$J_n(z) = \sum_{s=0}^{\infty}\frac{(-1)^s z^{n+2s}}{2^{n+2s}s!\,\Gamma(n+s+1)}.$$

Therefore
$$J_n(z) = \frac{\Gamma\left(\frac{1}{2}-n\right)}{2i\pi \,.\, 2^n \,.\, \Gamma\left(\frac{1}{2}\right)}\, y,$$

or
$$J_n(z) = \frac{\Gamma\left(\frac{1}{2}-n\right)}{2^n \Gamma\left(\frac{1}{2}\right)} \frac{z^n}{2i\pi} \int^{(1+,\,-1-)} e^{zti} (t^2-1)^{n-\frac{1}{2}}\, dt.$$

This formula gives the required expression of $J_n(z)$. It is valid for all values of n and of z; but when n is of the form $\left(k+\frac{1}{2}\right)$, where k is a positive integer, the factor $\Gamma\left(\frac{1}{2}-n\right)$ becomes infinite and the integral

$$\int^{(1+,\,-1-)} e^{zti} (t^2-1)^{n-\frac{1}{2}}\, dt$$

becomes zero (since the integrand is now regular at all points within the contour), so that for this exceptional case the formula is indeterminate.

Example. Deduce the formula

$$J_n(z) = \frac{z^n}{2^n\, \Gamma\left(\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(z\cos\phi) \sin^{2n}\phi\, d\phi$$

from the result of this article.

158. *Bessel functions as a limiting case of Legendre functions.*

We have already (§ 148) shewn that Bessel's differential equation of order m is the same as the differential equation of the associated Legendre functions

$$\operatorname*{Limit}_{n=\infty} P_n^m\left(1-\frac{z^2}{2n^2}\right) \text{ and } \operatorname*{Limit}_{n=\infty} Q_n^m\left(1-\frac{z^2}{2n^2}\right).$$

We shall now express this connexion more precisely, by establishing the formula

$$J_m(z) = \operatorname*{Limit}_{n=\infty} n^{-m} P_n^m\left(1-\frac{z^2}{2n^2}\right).$$

For taking the expression of the associated Legendre function by a definite integral (§ 131), we have

$$n^{-m} P_n^m\left(1-\frac{z^2}{2n^2}\right) = \frac{(n+m)(n+m-1)\ldots\ldots(n-m+1)\, z^m}{(2m-1)(2m-3)\ldots\ldots 1 \,.\, \pi \,.\, n^{2m}} \left(1-\frac{z^2}{4n^2}\right)^{\frac{1}{2}m}$$
$$\times \int_0^\pi \left\{1-\frac{z^2}{2n^2}+\cos\phi\left(-\frac{z^2}{n^2}+\frac{z^4}{4n^4}\right)\right\}^{n-m} \sin^{2m}\phi\, d\phi,$$

and as n becomes infinitely great, the right-hand side of this equation tends to the limiting value

$$\frac{z^m}{(2m-1)(2m-3)\ldots 1 \,.\, \pi} \int_0^\pi \left(1+\frac{iz}{n}\cos\phi\right)^n \sin^{2m}\phi\, d\phi,$$

or
$$\frac{z^m}{(2m-1)(2m-3)\dots 1 \,.\, \pi}\int_0^\pi e^{iz\cos\phi}\sin^{2m}\phi\,d\phi,$$

or (§ 155)
$$J_m(z),$$

which establishes the result stated; it is due to Heine*.

159. *Bessel functions whose order is half an odd integer.*

The result of § 157 suggests that when the order n of a Bessel function $J_n(z)$ is a number of the form $k+\frac{1}{2}$, where k is a positive integer, certain exceptional circumstances arise† in connexion with the function. In this case it is in fact possible to express the Bessel function

$$J_{k+\frac{1}{2}}(z)=\frac{z^{k+\frac{1}{2}}}{2^{k+\frac{1}{2}}\Gamma\left(k+\frac{3}{2}\right)}\left\{1-\frac{z^2}{2(2k+3)}+\frac{z^4}{2\,.\,4\,.\,(2k+3)(2k+5)}-\dots\right\}$$

in terms of well-known elementary functions.

For by § 151 we have, if k be a positive integer,

$$J_{k+\frac{1}{2}}(z)=(-2)^k z^{k+\frac{1}{2}}\frac{d^k}{d(z^2)^k}\left\{\frac{J_{\frac{1}{2}}(z)}{z^{\frac{1}{2}}}\right\}.$$

But the series-expansion of the function $J_{\frac{1}{2}}(z)$ is

$$J_{\frac{1}{2}}(z)=\frac{2^{\frac{1}{2}}z^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\left\{1-\frac{z^2}{2\,.\,3}+\frac{z^4}{2\,.\,3\,.\,4\,.\,5}-\dots\right\}=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z.$$

Therefore
$$J_{k+\frac{1}{2}}(z)=\frac{(-1)^k(2z)^{k+\frac{1}{2}}}{\pi^{\frac{1}{2}}}\frac{d^k}{d(z^2)^k}\left(\frac{\sin z}{z}\right),$$

which is the required expression of the function $J_{k+\frac{1}{2}}(z)$ in terms of more elementary functions.

The student will without difficulty be able to prove that a second solution of Bessel's differential equation in this case is

$$z^{k+\frac{1}{2}}\frac{d^k}{d(z^2)^k}\left(\frac{\cos z}{z}\right).$$

* Heine's definition of the associated Legendre function is somewhat different from that which has since become general and which is adopted in this book: this leads to differences of statement in many other formulae, such as that of this article.

† The student who is familiar with the theory of linear differential equations will observe that in this case, and also in the other exceptional case of n an integer, the difference of the roots of the "indicial equation" of Bessel's equation is an integer.

Example. Shew that the solution of the equation

$$z^{m+\frac{1}{2}} \frac{d^{2m+1}y}{dz^{2m+1}} + y = 0$$

is

$$y = z^{\frac{2m+1}{4}} \sum_{p=0}^{p=2m} c_p \{J_{-m-\frac{1}{2}}(2a_p z^{\frac{1}{2}}) + iJ_{m+\frac{1}{2}}(2a_p z^{\frac{1}{2}})\},$$

where the quantities c_p are arbitrary constants, and $a_0, a_1, \ldots a_{2m}$, are the roots of the equation

$$a^{2m+1} = i. \qquad \text{(Lommel.)}$$

160. *Expression of $J_n(z)$ in a form which furnishes an approximate value to $J_n(z)$ for large real positive values of z.*

We now proceed to form an integral which will be found to play the same part in the theory of the function $J_n(z)$ as the integral of § 104 plays in the theory of the function $\Gamma(z)$. We shall suppose z to be real and positive. Then, by § 155, we have, for all positive values of n,

$$J_n(z) = \frac{z^n}{2^n \cdot \Gamma\left(n+\frac{1}{2}\right) \cdot \pi^{\frac{1}{2}}} \int_0^\pi \cos(z\cos\phi)\sin^{2n}\phi\, d\phi.$$

Writing $\cos\phi = x$, this becomes

$$J_n(z) = \frac{z^n}{2^n \cdot \Gamma\left(n+\frac{1}{2}\right) \cdot \pi^{\frac{1}{2}}} \int_{-1}^{+1} (1-x^2)^{n-\frac{1}{2}} \cos zx\, dx,$$

or $$J_n(z) = \text{Real part of } \frac{z^n}{2^{n-1} \cdot \Gamma\left(n+\frac{1}{2}\right) \cdot \pi^{\frac{1}{2}}} \int_0^1 (1-x^2)^{n-\frac{1}{2}} e^{izx}\, dx.$$

In order to transform this integral, we take in the plane of a complex variable t a contour $OPQBCO$, formed in the following way. O is the origin $(t=0)$; P is the point $t = 1-\rho$, where ρ is a small quantity, and OP is the part of the real axis between O and P. Q is the point $t = 1 + i\rho$, and PQ is a quadrant of a circle which has its centre at the point $t=1$. B is the point $t = 1 + ik$, where k is a large positive quantity, and QB is the line (parallel to the imaginary axis in the t-plane) joining Q and B. C is the point $t = ik$, and BC is the line (parallel to the real axis) joining B and C. Lastly, CO is the part of the imaginary axis between C and O. Then the function

$$(1-t^2)^{n-\frac{1}{2}} e^{izt}$$

is regular at all points of the t-plane in the interior of the contour $OPQBCO$; and therefore the integral

$$\int (1-t^2)^{n-\frac{1}{2}} e^{izt}\, dt,$$

taken round this contour, is zero.

We can write this relation in the form

$$\int_{OP} + \int_{PQ} + \int_{QB} + \int_{BC} + \int_{CO} = 0.$$

Now the part of the integral due to PQ tends to zero with ρ, and the part due to BC tends to zero as k becomes infinitely great, while the part due to CO is purely imaginary. Thus we have

$$\text{Real part of} \int_{OP} = - \text{Real part of} \int_{QB},$$

and so $$J_n(z) = \text{Real part of} \frac{-z^n}{2^{n-1} \cdot \Gamma\left(n+\frac{1}{2}\right) \cdot \pi^{\frac{1}{2}}} \int_{QB} (1-t^2)^{n-\frac{1}{2}} e^{izt}\, dt.$$

In this integral write

$$t = 1 + \frac{iu}{z},$$

so that u varies between the limits 0 and ∞ when t describes the line QB;

and $$(1-t^2)^{\frac{1}{2}} = e^{-\frac{i\pi}{4}} \left(\frac{2}{z}\right)^{\frac{1}{2}} u^{\frac{1}{2}} \left(1 + \frac{iu}{2z}\right)^{\frac{1}{2}},$$

and therefore $$\int_{QB} = 2^{n-\frac{1}{2}}\, i e^{i\left\{z-(2n-1)\frac{\pi}{4}\right\}} z^{-n-\frac{1}{2}} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left(1+\frac{iu}{2z}\right)^{n-\frac{1}{2}} du.$$

Thus we have

$$J_n(z) = \text{Real part of} \frac{-2^{\frac{1}{2}}\, i e^{i\left\{z-(2n-1)\frac{\pi}{4}\right\}}}{\Gamma\left(n+\frac{1}{2}\right) \cdot \pi^{\frac{1}{2}} \cdot z^{\frac{1}{2}}} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left(1+\frac{iu}{2z}\right)^{n-\frac{1}{2}} du,$$

or $$J_n(z) = \frac{1}{\Gamma\left(n+\frac{1}{2}\right) \cdot (2\pi z)^{\frac{1}{2}}}$$

$$\times \left[\begin{array}{l} \cos\left\{z - \left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left\{\left(1+\frac{iu}{2z}\right)^{n-\frac{1}{2}} + \left(1-\frac{iu}{2z}\right)^{n-\frac{1}{2}}\right\} du \\ + \sin\left\{z - \left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left\{\left(1+\frac{iu}{2z}\right)^{n-\frac{1}{2}} - \left(1-\frac{iu}{2z}\right)^{n-\frac{1}{2}}\right\} i\, du \end{array}\right].$$

This is the integral-expression required. It is easily seen to furnish an approximate value of $J_n(z)$ for large positive values of z; for as z becomes indefinitely large, the two integrals in the expression tend respectively to the limits $2\Gamma\left(n+\frac{1}{2}\right)$ and zero; and therefore the function $J_n(z)$ approximates for large positive values of z to the value

$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left\{z - \left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\}.$$

The evaluation of $J_n(z)$ when z is large will be considered in fuller detail in the next article.

The result of this article can also be obtained in the following quite different manner, which connects it more closely with the general theory. We have seen in § 148 that Bessel's differential equation is a limiting case of the general hypergeometric equation, represented by the function

$$e^{iz} \operatorname*{Limit}_{c=\infty} P \begin{Bmatrix} 0 & \infty & c & \\ -n & \tfrac{1}{2} & 0 & z \\ n & \tfrac{3}{2}-2ic & 2ic-1 & \end{Bmatrix}.$$

Since the differential equation of the P-function

$$P \begin{Bmatrix} 0 & \infty & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

is (§ 142) satisfied by the integral

$$z^{\alpha}(z-c)^{\gamma}\int t^{\beta+\gamma+\alpha'-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}\,dt,$$

taken between suitable limits, we see that Bessel's equation is satisfied by the expression

$$\operatorname*{Limit}_{c=\infty} e^{iz}z^{-n}\int t^{n-\frac{1}{2}}\left(1-\frac{t}{c}\right)^{-n-\frac{3}{2}+2ic}(z-t)^{n-\frac{1}{2}}\,dt,$$

or

$$e^{iz}z^{-n}\int t^{n-\frac{1}{2}}e^{-2it}(z-t)^{n-\frac{1}{2}}\,dt,$$

or (putting $t=-ivz$)

$$e^{iz}z^{-n}\int v^{n-\frac{1}{2}}z^{n-\frac{1}{2}}(z+ivz)^{n-\frac{1}{2}}e^{-2vz}z\,dv,$$

or

$$e^{iz}z^{n}\int (v+iv^2)^{n-\frac{1}{2}}e^{-2vz}\,dv.$$

The limits of the integral can be taken to be 0 and ∞, since these satisfy the conditions for the limits found in § 142; and hence it follows that

$$e^{iz}z^{n}\int_0^{\infty}(v+iv^2)^{n-\frac{1}{2}}e^{-2vz}\,dv$$

is a solution of Bessel's equation.

Similarly the quantity

$$e^{-iz}z^{n}\int_0^{\infty}(v-iv^2)^{n-\frac{1}{2}}e^{-2vz}\,dv$$

is a solution of Bessel's equation.

The solution $J_n(z)$ must therefore be of the form

$$J_n(z)=Ae^{iz}z^{n}\int_0^{\infty}(v+iv^2)^{n-\frac{1}{2}}e^{-2vz}\,dv+Be^{-iz}z^{n}\int_0^{\infty}(v-iv^2)^{n-\frac{1}{2}}e^{-2vz}\,dv,$$

where A and B are constants independent of z. This is substantially the form given above, but the determination of the constants A and B is a matter of some difficulty, for which the student is referred to a memoir by Schafheitlin, *Crelle's Journal*, CXIV. p. 39.

Example. Shew (by making the substitution $u = 2z \cot \phi$ in the integral found above, or otherwise) that

$$J_n(z) = -\frac{2^{n+1} z^n}{\Gamma(n+\frac{1}{2}) \cdot \pi^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} e^{-2z\cot\phi} \cos^{n-\frac{1}{2}} \phi \operatorname{cosec}^{2n+1} \phi \cos\{z - (n-\tfrac{1}{2})\phi\}\, d\phi.$$

161. *The Asymptotic Expansion of the Bessel functions.*

The Bessel functions can for large values of the argument be represented by asymptotic expansions. We shall here consider only the asymptotic expansion of $J_n(z)$ for positive real values of z; this was discovered by Poisson (for $n = 0$) and Jacobi (for general integer values of n). The theorem has been considered for complex values of z by Hankel* and several subsequent writers.

We shall derive the asymptotic expansion from the integral-expression

$$J_n(z) = \frac{1}{(2\pi z)^{\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right)}$$
$$\times \left[\begin{aligned} &\cos\left\{z - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right\} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left\{\left(1 + \frac{iu}{2z}\right)^{n-\frac{1}{2}} + \left(1 - \frac{iu}{2z}\right)^{n-\frac{1}{2}}\right\} du \\ &+ \sin\left\{z - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right\} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left\{\left(1 + \frac{iu}{2z}\right)^{n-\frac{1}{2}} - \left(1 - \frac{iu}{2z}\right)^{n-\frac{1}{2}}\right\} i\, du \end{aligned} \right],$$

found in the last article.

It is first necessary to find the asymptotic expansion of the integral

$$\int_0^\infty e^{-u} u^k \left(1 + \frac{iu}{2z}\right)^k du, \quad (k > 0),$$

which we shall denote by the symbol I.

Now we have

$$\left(1 + \frac{iu}{2z}\right)^k = 1 + k\frac{iu}{2z} + \frac{k(k-1)}{2!}\left(\frac{iu}{2z}\right)^2 + \ldots + \frac{k(k-1)\ldots(k-n+1)}{n!}\left(\frac{iu}{2z}\right)^n$$
$$+ \frac{k(k-1)\ldots(k-n)}{n!} \int_0^{\frac{iu}{2z}} \left(\frac{iu}{2z} - t\right)^n (1+t)^{k-n-1} dt.$$

* *Math. Ann.* I.

Therefore

$$I=\int_0^\infty e^{-u}u^k\,du+\sum_{r=1}^{n}\frac{k(k-1)\dots(k-r+1)}{r!}\left(\frac{i}{2z}\right)^r\int_0^\infty e^{-u}u^{k+r}\,du$$

$$+\frac{k(k-1)\dots(k-n)}{n!}\int_0^\infty e^{-u}u^k\,du\int_0^{\frac{iu}{2z}}\left(\frac{iu}{2z}-t\right)^n(1+t)^{k-n-1}\,dt,$$

or

$$I=\Gamma(k+1)\left\{1+\sum_{r=1}^{n}\frac{k(k-1)\dots(k-r+1)}{r!}\left(\frac{i}{2z}\right)^r(k+r)(k+r-1)\dots(k+1)\right\}+R_n,$$

where

$$R_n=\frac{k(k-1)\dots(k-n)}{n!}\left(\frac{i}{2z}\right)^{n+1}\int_0^\infty e^{-u}u^k\,du\int_0^u(u-v)^n\left(1+\frac{iv}{2z}\right)^{k-n-1}dv.$$

Now as z becomes infinitely large, n having any definite finite integer value, the remainder-term R_n tends to the limit

$$R_n=\frac{k(k-1)\dots(k-n)}{n!}\left(\frac{i}{2z}\right)^{n+1}\int_0^\infty e^{-u}u^k\,du\int_0^u(u-v)^n\,dv,$$

or $$R_n=\frac{k(k-1)\dots(k-n)}{(n+1)!}\left(\frac{i}{2z}\right)^{n+1}\int_0^\infty e^{-u}u^{k-n+1}\,du,$$

or $$R_n=\frac{k(k-1)\dots(k-n)\,\Gamma(k-n+2)}{(n+1)!}\left(\frac{i}{2z}\right)^{n+1}.$$

It follows from this that

$$\operatorname*{Limit}_{z=\infty} z^nR_n=0,$$

and therefore the series

$$\Gamma(k+1)\left\{1+\sum_{r=1}^{\infty}\frac{(k+r)(k+r-1)\dots(k-r+1)}{r!}\left(\frac{i}{2z}\right)^r\right\}$$

is the asymptotic expansion of the function

$$\int_0^\infty e^{-u}u^k\left(1+\frac{iu}{2z}\right)^k du \qquad (k>0).$$

Substituting this result in the expression already found for $J_n(z)$, we see that

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2\pi z)^{\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}$$

$$\left[\begin{aligned}&\cos\left\{z-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\}\left\{2+\sum_{r=1}^{\infty}\frac{\left(n-\frac{1}{2}+r\right)\left(n-\frac{3}{2}+r\right)\dots\left(n+\frac{1}{2}-r\right)}{r!}\frac{i^r+(-i)^r}{(2z)^r}\right\}\\&+\sin\left\{z-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\}\left\{\sum_{r=1}^{\infty}\frac{\left(n-\frac{1}{2}+r\right)\left(n-\frac{3}{2}+r\right)\dots\left(n-r+\frac{1}{2}\right)}{r!}\frac{i^{r+1}-i\,.\,(-i)^r}{(2z)^r}\right\}\end{aligned}\right]$$

or $\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}$

$$\left[\cos\left\{z-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\}\left\{1+\sum_{r=1}^{\infty}\frac{\left(n-\frac{1}{2}+2r\right)\left(n-\frac{3}{2}+2r\right)\dots\left(n-2r+\frac{1}{2}\right)}{2r!}\frac{(-1)^r}{(2z)^{2r}}\right\}\right.$$
$$\left.+\sin\left\{z-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\}\sum_{r=1}^{\infty}\frac{\left(n-\frac{3}{2}+2r\right)\left(n-\frac{5}{2}+2r\right)\dots\left(n-2r+\frac{3}{2}\right)}{(2r-1)!}\frac{(-1)^r}{(2z)^{2r-1}}\right]$$

is the asymptotic expansion of the Bessel function $J_n(z)$ for large positive values of z.

Even when z is not very large, the value of $J_n(z)$ can be computed with great accuracy from this formula. Thus for all values of z greater than 8, the first three terms of this asymptotic expansion give the value of $J_0(z)$ and $J_1(z)$ correct to six places of decimals.

162. *The second solution of Bessel's equation when the order is an integer.*

We have seen in § 149 that when the order n of Bessel's differential equation is not an integer, the general solution of the equation is

$$\alpha J_n(z)+\beta J_{-n}(z),$$

where α and β are arbitrary constants.

When however n is an integer, we have seen that

$$J_n(z)=(-1)^n J_{-n}(z),$$

and consequently the two solutions $J_n(z)$ and $J_{-n}(z)$ are not really distinct. We therefore require in this case to find another particular solution of the differential equation, distinct from $J_n(z)$, in order to have the general solution.

To obtain this second solution, we write

$$y=uJ_n(z),$$

where u is a new dependent variable, in Bessel's equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0.$$

Remembering that $J_n(z)$ is a solution of Bessel's equation, the differential equation for u becomes

$$J_n(z)\frac{d^2u}{dz^2}+\left\{2\frac{dJ_n(z)}{dz}+\frac{1}{z}J_n(z)\right\}\frac{du}{dz}=0,$$

or
$$\frac{\dfrac{d^2u}{dz^2}}{\dfrac{du}{dz}}+2\frac{\dfrac{dJ_n(z)}{dz}}{J_n(z)}+\frac{1}{z}=0.$$

Integrating this equation, we have

$$\log\frac{du}{dz}+2\log J_n(z)+\log z=\text{constant},$$

or
$$\frac{du}{dz}=\frac{b}{z\{J_n(z)\}^2},\ \text{where } b \text{ is a constant,}$$

or
$$u=a+b\int^z\frac{dt}{t\{J_n(t)\}^2},$$

where a and b are arbitrary constants.

The complete solution of Bessel's equation can therefore be written in the form

$$y=aJ_n(z)+bJ_n(z)\int^z\frac{dt}{t\{J_n(t)\}^2}.$$

To find the nature of the solution thus obtained, we observe that in the vicinity of the point $t=0$ the integrand

$$t^{-1}\{J_n(t)\}^{-2}$$

is of the form

$$t^{-2n-1}(\text{constant} + \text{powers of } t^2)^{-2},$$

which when n is a positive integer can be expanded as a Laurent series in the form

$$\frac{c_{-2n-1}}{t^{2n+1}}+\frac{c_{-2n+1}}{t^{2n-1}}+\ldots+\frac{c_{-3}}{t^3}+\frac{c_{-1}}{t}+c_1t+\ldots.$$

The function

$$\int^z t^{-1}\{J_n(t)\}^{-2}\,dt$$

has therefore the form

$$\frac{d_{-2n}}{z^{2n}}+\frac{d_{-2n+2}}{z^{2n-2}}+\ldots+\frac{d_{-2}}{z^2}+d\log z+d_2z^2+\ldots,$$

where the quantities d_{-2n}, d_{-2n+2}, ... are definite constants.

It thus appears that the complete solution of Bessel's equation can be written in the form

$$y=AJ_n(z)+B\{J_n(z)\log z+v\},$$

where v is the result obtained by multiplying together $J_n(z)$ and a Laurent series of the form

$$\frac{d_{-2n}}{z^{2n}}+\frac{d_{-2n+2}}{z^{2n-2}}+\ldots+\frac{d_{-2}}{z^2}+d_2z^2+\ldots,$$

where the quantities d_{-2n}, d_{-2n+2}, ... are definite constants, and A and B are arbitrary constants. The expansion of $J_n(z)$ being known, we see that the product v has the form

$$z^{-n} \times \text{a power-series in } z^2;$$

and thus *a second solution of Bessel's differential equation, in the case in which n is an integer, can be taken of the form*

$$J_n(z) \log z + z^{-n}(a_0 + a_1 z^2 + a_2 z^4 + a_3 z^6 + \ldots),$$

where the quantities $a_0, a_1, a_2, \ldots$ are definite constants. These quantities a are not however all of them strictly speaking definite, since by adding a multiple of $J_n(z)$ (which will leave the expression still a solution of Bessel's equation), it is possible to change all the quantities a after a_{n-1}.

*This solution will be denoted by $K_n(z)$**.

The coefficients $a_0, a_1, a_2, \ldots$ may theoretically be determined by substituting this expansion in the differential equation, and equating to zero the coefficients of successive powers of z. A better method is however the following†.

We have seen that when n is a positive integer, $J_{-n}(z)$ reduces to $(-1)^n J_n(z)$; in fact, if in the equation

$$J_{-(n-\epsilon)}(z) = \left(\frac{z}{2}\right)^{-(n-\epsilon)} \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(-n+\epsilon+p+1)\,\Gamma(p+1)} \left(\frac{z}{2}\right)^{2p}$$

we suppose the quantity ϵ to tend to zero, all the terms of the series vanish as far as $p=n$, since $\Gamma(-n+p+1)$ is for these terms infinite. Changing the meaning of the index of summation p in the other terms, we have

$$J_{-(n-\epsilon)}(z) = \left(\frac{z}{2}\right)^{-n+\epsilon} \sum_{p=0}^{n-1} \frac{(-1)^p}{\Gamma(-n+\epsilon+p+1)\,\Gamma(p+1)} \left(\frac{z}{2}\right)^{2p}$$
$$+ (-1)^n \left(\frac{z}{2}\right)^{n+\epsilon} \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(\epsilon+p+1)\,\Gamma(n+p+1)} \left(\frac{z}{2}\right)^{2p},$$

and when $\epsilon = 0$, the first of these partial series is zero and the second is

$$(-1)^n J_n(z).$$

Since the quantity

$$(-1)^n J_{-(n-\epsilon)}(z) - J_{(n-\epsilon)}(z)$$

vanishes with ϵ, we can take as a second solution of Bessel's equation the limiting value of the quotient

$$\frac{(-1)^n J_{-(n-\epsilon)}(z) - J_{(n-\epsilon)}(z)}{\epsilon}.$$

* In referring to memoirs it must be borne in mind that different writers have taken different definitions of the Bessel functions of the second kind.

† Due to Hankel, *Math. Ann.* I. p. 470 (1869).

Substituting the above values for the Bessel functions, this becomes

$$\left(\frac{z}{2}\right)^{-(n-\epsilon)}\sum_{p=0}^{n-1}\frac{(-1)^{p+n}}{\Gamma(p+1)}\frac{1}{\epsilon\Gamma(-n+\epsilon+p+1)}\left(\frac{z}{2}\right)^{2p}+\left(\frac{z}{2}\right)^{n}\sum_{p=0}^{\infty}(-1)^p\frac{f(\epsilon)}{\epsilon}\left(\frac{z}{2}\right)^{2p},$$

where $f(\epsilon)$ represents the expression

$$f(\epsilon)=\frac{1}{\Gamma(n+p+1)\Gamma(p+\epsilon+1)}\left(\frac{z}{2}\right)^{\epsilon}-\frac{1}{\Gamma(n+p-\epsilon+1)\Gamma(p+1)}\left(\frac{z}{2}\right)^{-\epsilon}.$$

The limiting value of $f(\epsilon)/\epsilon$, as ϵ tends to zero, is

$$\frac{1}{\Gamma(n+p+1)\Gamma(p+1)}2\log\frac{z}{2}-\frac{1}{\Gamma(n+p+1)}\frac{\Gamma'(p+1)}{\{\Gamma(p+1)\}^2}-\frac{1}{\Gamma(p+1)}\frac{\Gamma'(n+p+1)}{\{\Gamma(n+p+1)\}^2}.$$

Also, since

$$\frac{1}{\Gamma(-n+\epsilon+p+1)}=\frac{1}{\pi}\Gamma(n-\epsilon-p)\sin(-n+\epsilon+p+1)\pi,$$

we have $$\operatorname*{Limit}_{\epsilon=0}\frac{1}{\epsilon\Gamma(-n+\epsilon+p+1)}=(-1)^{n+p+1}\Gamma(n-p).$$

Consequently we obtain, as a second particular solution of Bessel's equation, the expression *

$$-\left(\frac{z}{2}\right)^{-n}\sum_{p=0}^{n-1}\frac{\Gamma(n-p)}{\Gamma(p+1)}\left(\frac{z}{2}\right)^{2p}+\left(\frac{z}{2}\right)^{n}\sum_{p=0}^{\infty}\frac{(-1)^p}{\Gamma(n+p+1)\Gamma(p+1)}\left(\frac{z}{2}\right)^{2p}\left\{2\log\frac{z}{2}-\frac{\Gamma'(n+p+1)}{\Gamma(n+p+1)}-\frac{\Gamma'(p+1)}{\Gamma(p+1)}\right\}.$$

The coefficient of $\log z$ in this expression is $2J_n(z)$. So, dividing the expression by 2, we have the second solution in the form

$$J_n(z)\log z-\frac{z^{-n}}{2^{-n+1}}\sum_{p=0}^{n-1}\frac{\Gamma(n-p)}{\Gamma(p+1)}\left(\frac{z}{2}\right)^{2p}-J_n(z)\log 2$$
$$+\left(\frac{z}{2}\right)^{n}\sum_{p=0}^{\infty}\frac{(-1)^p}{\Gamma(n+p+1)\Gamma(p+1)}\left(\frac{z}{2}\right)^{2p}\frac{1}{2}\left\{-\frac{\Gamma'(n+p+1)}{\Gamma(n+p+1)}-\frac{\Gamma'(p+1)}{\Gamma(p+1)}\right\}.$$

It is convenient to add to this expression a term

$$J_n(z)\left\{\log 2+\frac{\Gamma'(1)}{\Gamma(1)}\right\},$$

* This is Hankel's second solution $Y_n(z)$. It is really

$$\frac{dJ_n(z)}{dn}+(-1)^{n+1}\frac{dJ_{-n}(z)}{dn}.$$

which is itself a solution of Bessel's equation; so the second solution now takes the form

$$J_n(z)\log z - \frac{1}{2}\sum_{p=0}^{n-1}\frac{(n-p-1)!}{p!}\left(\frac{z}{2}\right)^{2p-n}$$

$$-\frac{1}{2}\left(\frac{z}{2}\right)^n\sum_{p=0}^{\infty}\frac{(-1)^p}{\Gamma(n+p+1)\,\Gamma(p+1)}\left(\frac{z}{2}\right)^{2p}\left\{\frac{\Gamma'(n+p+1)}{\Gamma(n+p+1)}+\frac{\Gamma'(p+1)}{\Gamma(p+1)}-\frac{2\Gamma'(1)}{\Gamma(1)}\right\}.$$

This is the solution $K_n(z)$ which we take as our standard.

Since, when r is a positive integer, we have

$$\frac{\Gamma'(r+1)}{\Gamma(r+1)}-\frac{\Gamma'(1)}{\Gamma(1)}=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{r},$$

we can write $K_n(z)$ in the form

$$K_n(z)=J_n(z)\log z-\frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{p=0}^{n-1}\frac{(n-p-1)!}{p!}\left(\frac{z}{2}\right)^{2p}$$

$$-\frac{1}{2}\left(\frac{z}{2}\right)^n\sum_{p=0}^{\infty}\frac{(-1)^p}{(n+p)!\,p!}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p}+1+\frac{1}{2}+\dots+\frac{1}{n+p}\right\}\left(\frac{z}{2}\right)^{2p}.$$

When n is an integer, the two independent solutions of Bessel's differential equation are $J_n(z)$ and $K_n(z)$.

Example 1. Shew that the function $K_n(z)$ satisfies the recurrence-formulae

$$nK_n(z)=\tfrac{1}{2}z\{K_{n+1}(z)+K_{n-1}(z)\},$$

$$\frac{dK_n(z)}{dz}=\tfrac{1}{2}\{K_{n-1}(z)-K_{n+1}(z)\}.$$

These are the same as the recurrence-formulae satisfied by $J_n(z)$.

Example 2. When the real part of z is positive, shew that the expression

$$\int_0^{\pi}\sin(z\sin\phi-n\phi)\,d\phi-\int_0^{\infty}e^{-z\sinh\theta}\{e^{n\theta}+(-1)^n e^{-n\theta}\}\,d\theta$$

is a second solution of Bessel's differential equation of integer order n.

(Schläfli.)

Example 3. Shew that the expression

$$J_0\log z+2\,(J_2-\tfrac{1}{2}J_4+\tfrac{1}{3}J_6-\dots)$$

is a second solution of the Bessel equation of order zero.

163. *Neumann's expansion; determination of the coefficients.*

We shall now consider* the expansion of an arbitrary function $f(z)$, regular at the origin, in a series of Bessel functions, in the form

$$f(z) = \alpha_0 J_0(z) + \alpha_1 J_1(z) + \alpha_2 J_2(z) + \dots,$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2, \dots$ are independent of z.

Suppose first that such an expansion is possible, and let us try to determine the coefficients, by expanding both sides of the equation as power-series in z and equating coefficients of the several powers of z. Since

$$f(z) = f(0) + 2\left(\frac{z}{2}\right) f'(0) + \frac{2^2}{2!}\left(\frac{z}{2}\right)^2 f''(0) + \frac{2^3}{3!}\left(\frac{z}{2}\right)^3 f'''(0) + \dots$$

and $$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \left\{1 - \frac{1}{1!(n+1)}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(n+1)(n+2)}\left(\frac{z}{2}\right)^4 - \dots\right\},$$

we have on comparing coefficients the equalities

$$f(0) = \alpha_0,$$
$$2f'(0) = \alpha_1,$$
$$2^2 f''(0) = -2\alpha_0 + \alpha_2, \text{ etc.},$$

from which without difficulty we find

$$\alpha_0 = f(0),$$

$$\alpha_n = 2\left\{f(0) + \frac{n^2}{2^2} f''(0) + \frac{n^2(n^2-2^2)}{4!} f^{\text{iv}}(0) + \dots + 2^{n-1} f^{(n)}(0)\right\} \quad (n \text{ even}),$$

$$\alpha_n = 2\left\{nf'(0) + \frac{n(n^2-1^2)}{3!} f'''(0) + \frac{n(n^2-1^2)(n^2-3^2)}{5!} f^{(\text{v})}(0) + \dots + 2^{n-1} f^{(n)}(0)\right\} \quad (n \text{ odd}).$$

These coefficients take a simpler form, if we introduce functions $O_1(z)$, $O_2(z)$, $O_3(z)$, ..., defined by the formulae

$$O_n(z) = \frac{1}{z} + \frac{n^2}{z^3} + \frac{n^2(n^2-2^2)}{z^5} + \dots + \frac{2^{n-1} n!}{z^{n+1}} \quad (n \text{ even}),$$

$$O_n(z) = \frac{n}{z^2} + \frac{n(n^2-1^2)}{z^4} + \frac{n(n^2-1^2)(n^2-3^2)}{z^6} + \dots + \frac{2^{n-1} n!}{z^{n+1}} \quad (n \text{ odd});$$

for then it is easily seen that α_n is twice the residue of the function $O_n(t) f(t)$

* C. Neumann, *Theorie der Bessel'schen Functionen.* The exposition here given follows Kapteyn, *Annales de l'École Normale* (3) x. p. 106 (1893).

at the point $t=0$. The two formulae for $O_n(z)$ can be united by reversing the order of the terms; thus

$$O_n(z)=\frac{2^{n-1}\,n!}{z^{n+1}}\left\{1+\frac{z^2}{2(2n-2)}+\frac{z^4}{2\,.\,4(2n-2)(2n-4)}+\dots\right\},$$

the series terminating with the term in z^n or z^{n-1}.

We thus have Neumann's expansion

$$f(z)=\alpha_0 J_0(z)+\alpha_1 J_1(z)+\alpha_2 J_2(z)+\dots,$$

where $$\alpha_0=f(0),$$

and α_n $(n>0)$ is twice the residue of $O_n(t)f(t)$ at the point $t=0$, so that

$$\alpha_n=\frac{1}{\pi i}\int_\gamma O_n(t)f(t)\,dt,$$

where γ is any simple contour surrounding the origin.

164. *Proof of Neumann's expansion.*

The method by which this result has been found cannot be regarded as a proof, since the possibility of the expansion was assumed. We can, however, now furnish a proof by determining directly the sum of the series obtained.

From the definition of $O_n(z)$, we can at once obtain the identities

$$O_{n+1}(z)+2\frac{dO_n(z)}{dz}-O_{n-1}(z)=0,\quad (n>0),$$

$$O_1(z)=-\frac{d}{dz}\left(\frac{1}{z}\right),$$

$$O_0(z)=\frac{1}{z}.$$

Writing the first of these equations in the symbolic form

$$O_{n+1}-2DO_n-O_{n-1}=0,\quad \text{where } D=\frac{d}{dz},$$

and solving the series of recurrence-equations obtained by giving n integer values, in the same way as if D were an algebraic quantity, we obtain for O_n the symbolic expression

$$O_n(z)=\tfrac{1}{2}\left[\{-D+(D^2+1)^{\frac{1}{2}}\}^n+\{-D-(D^2+1)^{\frac{1}{2}}\}^n\right]\left(\frac{1}{z}\right).$$

This symbolic expression can be transformed into a definite integral in the following way.

We have $$\frac{1}{t} = \int_0^\infty e^{-tu}\, du,$$

where the upper limit must be understood to mean that direction at infinity which makes the real part of tu positive and infinite; and therefore

$$O_n(t) = \int_0^\infty \tfrac{1}{2} e^{-tu} [\{u + (u^2+1)^{\frac{1}{2}}\}^n + \{u - (u^2+1)^{\frac{1}{2}}\}^n]\, du,$$

or, writing $tu = x$,

$$O_n(t) = \int_0^\infty \tfrac{1}{2} t^{-n-1} e^{-x} [\{x + (x^2+t^2)^{\frac{1}{2}}\}^n + \{x - (x^2+t^2)^{\frac{1}{2}}\}^n]\, dx,$$

where the upper limit now means the real positive infinity, so that the integration may be regarded as taken along the real axis of x.

Writing this in the form

$$O_n(t) = \underset{X=\infty}{\text{Limit}} \frac{1}{2t} \int_0^X \left[\left\{\frac{x+(x^2+t^2)^{\frac{1}{2}}}{t}\right\}^n + (-1)^n \left\{\frac{t}{x+(x^2+t^2)^{\frac{1}{2}}}\right\}^n\right] e^{-x}\, dx,$$

we have $$O_0(t)\, J_0(z) + 2 \sum_{n=1}^\infty O_n(t)\, J_n(z)$$

$$= \frac{1}{2t} \underset{X=\infty}{\text{Limit}} \int_0^X \sum_{n=0}^\infty \left\{\left[\left\{\frac{x+(x^2+t^2)^{\frac{1}{2}}}{t}\right\}^n + (-1)^n \left\{\frac{t}{x+(x^2+t^2)^{\frac{1}{2}}}\right\}^n\right] J_n(z)\right\} e^{-x}\, dx,$$

(by § 146)

$$= \frac{1}{2t} \underset{X=\infty}{\text{Limit}} \int_0^X 2e^{\frac{z}{2}\left\{\frac{x+(x^2+t^2)^{\frac{1}{2}}}{t} - \frac{t}{x+(x^2+t^2)^{\frac{1}{2}}}\right\}} . e^{-x}\, dx$$

$$= \frac{1}{t} \underset{X=\infty}{\text{Limit}} \int_0^X e^{\frac{zx}{t} - x}\, dx.$$

In order that this integral may have a meaning, the real part of $\frac{z-t}{t}$ must be negative, a condition which is fulfilled when

$$|z| < |t|.$$

If this inequality is satisfied, we have therefore

$$O_0(t)\, J_0(z) + 2 \sum_{n=1}^\infty O_n(t)\, J_n(z) = \frac{1}{t-z}.$$

From this result Neumann's expansion can at once be derived; for let $f(z)$ be any function which is regular in the interior of a circle C whose centre is at the origin, and let t be a point on the circumference of the circle. Then if z be any point in the interior of the circle, the condition $|z| < |t|$ is satisfied, and therefore we have

$$\frac{1}{t-z} = O_0(t)\, J_0(z) + 2 \sum_{n=1}^\infty O_n(t)\, J_n(z).$$

Thus $f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z}$

$$=\frac{1}{2\pi i}\int_C f(t)\,dt\left\{O_0(t)\,J_0(z)+2\sum_{n=1}^{\infty}O_n(t)\,J_n(z)\right\}$$

$$=\alpha_0 J_0(z)+\alpha_1 J_1(z)+\alpha_2 J_2(z)+\ldots,$$

where
$$\alpha_0=f(0)$$

and
$$\alpha_n=\frac{1}{\pi i}\int_C O_n(t)\,f(t)\,dt \qquad (n>0).$$

This establishes the validity of Neumann's expansion for points z within the circle C.

Example. Shew that

$$\cos z=J_0(z)-2J_2(z)+2J_4(z)-\ldots,$$

$$\sin z=2J_1(z)-2J_3(z)+2J_5(z)-\ldots.$$

165. *Schlömilch's expansion of an arbitrary function in terms of Bessel functions of order zero.*

Schlömilch * has given an expansion of a quite different character to that of Neumann. His result may be stated thus:

Any function $f(z)$ which is finite and continuous for real values of z between the limits $z=0$ and $z=\pi$, both inclusive, may be expressed in the form

$$f(z)=a_0+a_1 J_0(z)+a_2 J_0(2z)+a_3 J_0(3z)+\ldots,$$

where
$$a_0=f(0)+\frac{1}{\pi}\int_0^{\pi} u\int_0^1(1-t^2)^{-\frac{1}{2}}f'(ut)\,dt\,du,$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} u\cos nu\int_0^1(1-t^2)^{-\frac{1}{2}}f'(ut)\,dt\,du \qquad (n>0).$$

Schlömilch's proof is substantially as follows.

Suppose that F and f are two functions connected by the relation

$$f(z)=\frac{2}{\pi}\int_0^1(1-s^2)^{-\frac{1}{2}}F(zs)\,ds.$$

Then we have

$$f'(z)=\frac{2}{\pi}\int_0^1(1-s^2)^{-\frac{1}{2}}sF'(zs)\,ds.$$

* *Zeitschrift für Math. u. Physik*, II. (1857).

In this equation, write zt for z, multiply both sides by $z(1-t^2)^{-\frac{1}{2}}\,dt$, and integrate with respect to t between the limits $t=0$ and $t=1$. Thus

$$z\int_0^1 (1-t^2)^{-\frac{1}{2}} f'(zt)\,dt = \frac{2z}{\pi}\int_0^1 (1-t^2)^{-\frac{1}{2}}\,dt \int_0^1 (1-s^2)^{-\frac{1}{2}} sF'(zst)\,ds$$

$$= \frac{2}{\pi}\int_0^z \int_0^{(z^2-x^2)^{\frac{1}{2}}} (z^2-x^2-y^2)^{-\frac{1}{2}} F'(x)\,dx\,dy,$$

where $$x = zst, \quad y = zs(1-t^2)^{\frac{1}{2}}.$$

Performing the integrations, we have

$$z\int_0^1 (1-t^2)^{\frac{1}{2}} f'(zt)\,dt = F(z) - F(0).$$

Now by the definition of the function f, we have

$$f(0) = F(0).$$

Thus $$F(z) = f(0) + z\int_0^1 (1-t^2)^{-\frac{1}{2}} f'(zt)\,dt.$$

This equation expresses the function F explicitly in terms of the function f, whereas in the original definition f was expressed explicitly in terms of F.

In order to obtain Schlömilch's expansion, it is merely necessary to apply Fourier's theorem to the function $F(zs)$. We thus have

$$f(z) = \frac{2}{\pi}\int_0^1 (1-s^2)^{-\frac{1}{2}}\,ds \left\{\frac{1}{\pi}\int_0^\pi F(u)\,du + \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^\pi \cos nu \cos nzs\, F(u)\,du\right\}$$

$$= \frac{1}{\pi}\int_0^\pi F(u)\,du + \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^\pi \cos nu\, F(u)\, J_0(nz)\,du.$$

In this equation, replace $F(u)$ by its value in terms of $f(u)$. Thus we have

$$f(z) = \frac{1}{\pi}\int_0^\pi \left\{f(0) + u\int_0^1 (1-t^2)^{-\frac{1}{2}} f'(ut)\,dt\right\} du$$

$$+ \frac{2}{\pi}\sum_{n=1}^{\infty} J_0(nz)\int_0^\pi \cos nu \left\{f(0) + u\int_0^1 (1-t^2)^{-\frac{1}{2}} f'(ut)\,dt\right\} du,$$

which is Schlömilch's expansion.

Example. Shew that if $0 \leqslant z \leqslant \pi$, the expression

$$\frac{\pi^2}{4} - 2\left\{J_0(z) + \frac{1}{9}J_0(3z) + \frac{1}{25}J_0(5z) + \dots\right\}$$

is equal to z; but that, if $\pi \leqslant z \leqslant 2\pi$, its value is

$$z + 2\pi \cos^{-1}\frac{\pi}{z} - 2(z^2 - \pi^2),$$

where $\cos^{-1}\frac{\pi}{z}$ is taken between 0 and $\frac{\pi}{3}$.

Find the value of the expression when z lies between 2π and 3π.

(Cambridge Mathematical Tripos.)

166. *Tabulation of the Bessel functions.*

Many numerical tables of the Bessel functions have been published. Meissel's tables (Berlin, 1889) give the functions $J_0(z)$ and $J_1(z)$ to 12 decimal places for real values of z from $z=0$ to $z=15\frac{1}{2}$, at intervals of 0·01.

Tables of the second solution $Y_0(z)$, defined by the equation

$$Y_0(z) = J_0(z)\log z + \frac{z^2}{2^2} - \left(1+\frac{1}{2}\right)\frac{z^4}{2^2 \cdot 4^2} + \ldots,$$

from $z=0$ to $z=10{\cdot}2$, are given by B. A. Smith, *Messenger of Math.* XXVI. (1897).

The British Association Reports for 1889, 1893, 1896, contain tables of the functions $I_n(z)$, which are solutions of the differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} - \left(1+\frac{n^2}{z^2}\right)u = 0,$$

so that
$$I_n(z) = i^{-n} J_n(iz).$$

A table of the first 40 roots of $J_0(z)$ is given by Wilson and Peirce, *Bull. Amer. Math. Soc.* III. (1897).

Miscellaneous Examples.

1. Shew (e.g. by multiplying the expansions for $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$ and $e^{-\frac{1}{2}z\left(t-\frac{1}{t}\right)}$, and equating the terms independent of t) that

$$\{J_0(z)\}^2 + 2\{J_1(z)\}^2 + 2\{J_2(z)\}^2 + 2\{J_3(z)\}^2 + \ldots = 1,$$

and hence that, for real values of z, $J_0(z)$ can never exceed unity, and the other Bessel coefficients of higher order can never exceed $2^{-\frac{1}{2}}$.

2. Shew that, for all values of μ and ν,

$$J_\mu(z)J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{\mu+\nu+2n}{n} (\frac{1}{2}z)^{\mu+\nu+2n}}{\Gamma(\mu+n+1)\,\Gamma(\nu+n+1)}.$$

3. Shew that

$$J_3(z) = -3\frac{dJ_0(z)}{dz} + \frac{3}{z}\frac{d^2J_0(z)}{dz^2} - 2\frac{d^3J_0(z)}{dz^3}.$$

4. Shew that

$$\frac{J_{n+1}(z)}{J_n(z)} = \frac{\frac{1}{2}z}{n+1-}\;\frac{(\frac{1}{2}z)^2}{n+2-}\;\frac{(\frac{1}{2}z)^2}{n+3-\ldots}.$$

5. Shew that

$$J_{-\mu}(z)\,J_{\mu-1}(z) + J_{-\mu+1}(z)\,J_\mu(z) = \frac{2\sin\mu\pi}{\pi z}.$$

(Lommel.)

6. If $\dfrac{J_{n+1}(z)}{zJ_n(z)}$ be denoted by $Q_n(z)$, shew that

$$\frac{dQ_n(z)}{dz} = \frac{1}{z} - \frac{2(n+1)}{z}Q_n(z) + z\{Q_n(z)\}^2.$$

7. Shew that

$$\int_0^\pi J_{2r}(2z\cos\theta)\,d\theta = \pi\{J_r(z)\}^2.$$

8. If the function

$$\frac{1}{\pi}\int_0^\pi 2^k\cos^k u\cos(mu - z\sin u)\,du$$

(which when k is zero reduces to a Bessel function) be denoted by $J_m{}^k(z)$, shew that

$$J_m{}^k(z) = \sum_{p=0}^{\infty}\frac{1}{p!}(\tfrac{1}{2}z)^p N_{-m,k,p},$$

where $N_{-m,k,p}$ is the "Cauchy's number" defined by the equation

$$N_{-m,k,p} = \frac{1}{2\pi}\int_0^{2\pi} e^{-miu}(e^{iu}+e^{-iu})^k(e^{iu}-e^{-iu})^p\,du.$$

Shew further that this function satisfies the equations

$$J_m^k(z) = J_{m-1}^{k-1}(z) + J_{m+1}^{k-1}(z),$$

and
$$zJ_m^{k+2}(z) = 2mJ_m^{k+1}(z) - 2(k+1)\{J_{m-1}^k(z) - J_{m+1}^k(z)\}.$$

(Bourlet.)

9. If quantities v and M are connected by the equations

$$M = E - e\sin E,\quad \cos v = \frac{\cos E - e}{1 - e\cos E},\quad \text{where } |e| < 1,$$

shew that
$$v = M + 2(1-e^2)^{\frac{1}{2}}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(\tfrac{1}{2}e)^k J_m{}^k(me)\frac{1}{m}\sin mM,$$

where
$$J_m{}^k(z) = \frac{1}{\pi}\int_0^\pi (2\cos u)^k\cos(mu - z\sin u)\,du.$$

10. Prove that

$$P_n{}^m(\cos\theta) = \frac{c_n{}^m}{r^n}J_m\left\{(x^2+y^2)^{\frac{1}{2}}\frac{\partial}{\partial z}\right\}z^n,$$

where $z = r\cos\theta$, $x^2+y^2 = r^2\sin^2\theta$, and $c_n{}^m$ is a numerical quantity.

(Cambridge Mathematical Tripos, Part II, 1893.)

11. Shew that, if n is a positive integer and $(m+2n+1)$ is positive,

$$(m-1)\int_0^x x^m J_{n+1}(x)\,J_{n-1}(x)\,dx = x^{m+1}\{J_{n+1}(x)\,J_{n-1}(x) - J_n{}^2(x)\} + (m+1)\int_0^x x^m J_n{}^2(x)\,dx.$$

(Cambridge Mathematical Tripos, Part I, 1899.)

12. Prove that

$$J_0(z) = \frac{2}{\pi}\int_0^\infty \sin(z\cosh u)\,du.$$

(Cambridge Mathematical Tripos, Part II, 1893.)

13. Prove that

$$J_n(z) = \frac{z^n}{2^{n-1}\pi^{\frac{1}{2}}\,\Gamma(n+\frac{1}{2})}\left(1+\frac{d^2}{dz^2}\right)^{n-\frac{1}{2}}\left(\frac{\sin z}{z}\right),$$

and if $Y_n(z)$ is Hankel's second solution of Bessel's equation, defined by the equation

$$\frac{1}{\pi}Y_n(z) = \underset{n=\text{integer}}{\text{Limit}}\ \frac{J_{-n}(z) - J_n(z)\cos n\pi}{\sin n\pi},$$

shew that
$$\frac{1}{\pi}Y_n(z) = \frac{z^n}{2^{n-1}\pi^{\frac{1}{2}}\,\Gamma(n+\frac{1}{2})}\left(1+\frac{d^2}{dz^2}\right)^{n-\frac{1}{2}}\left(\frac{\cos z}{z}\right).$$

14. Shew how to express $z^{2n}J_{2n}(z)$ in the form

$$AJ_2(z) + BJ_0(z),$$

where A, B are polynomials in z; and prove that

$$J_4(6^{\frac{1}{2}}) + 3J_0(6^{\frac{1}{2}}) = 0,$$

$$3J_6(30^{\frac{1}{2}}) + 5J_2(30^{\frac{1}{2}}) = 0.$$

(Cambridge Mathematical Tripos, Part II, 1896.)

15. Prove that, if $J_n(\alpha\xi)=0$ and $J_n(\beta\xi)=0$,

$$\int_0^\xi xJ_n(\alpha x)\,J_n(\beta x)\,dx = 0, \text{ and } \int_0^\xi x\{J_n(\alpha x)\}^2\,dx = \tfrac{1}{2}\xi^2\{J_{n+1}(\alpha\xi)\}^2.$$

Hence prove that the roots of $J_n(x)=0$, other than zero, are all real and unequal.

(Cambridge Mathematical Tripos, Part I, 1893.)

16. Shew that

$$\int_0^\infty x^{-n+m}J_n(ax)\,dx = 2^{-n+m}\,a^{n-m-1}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(n-\frac{m-1}{2}\right)},$$

if
$$2n+1 > m > -1.$$

(Cambridge Mathematical Tripos, Part I, 1898.)

17. Shew that

$$\frac{z}{\pi}=\sum_{p=0}^{\infty}\frac{2p+1}{2}\{J_{\frac{2p+1}{2}}(z)\}^2.$$

(Lommel.)

18. Shew that the solution of the differential equation

$$\frac{d^2y}{dz^2}-\frac{\phi'}{\phi}\frac{dy}{dz}+\left\{\frac{1}{4}\left(\frac{\phi'}{\phi}\right)^2-\frac{1}{2}\frac{d}{dz}\left(\frac{\phi'}{\phi}\right)-\frac{1}{4}\left(\frac{\psi''}{\psi'}\right)^2+\frac{1}{2}\frac{d}{dz}\left(\frac{\psi''}{\psi'}\right)+\left(\psi^2-\frac{4\nu^2-1}{4}\right)\left(\frac{\psi'}{\psi}\right)^2\right\}y=0,$$

where ϕ and ψ are arbitrary functions of z, is

$$y=\left(\frac{\phi\psi}{\psi'}\right)^{\frac{1}{2}}\{AJ_\nu(\psi)+BJ_{-\nu}(\psi)\}.$$

19. Shew that

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}J_m(z\sin\theta)\sin^{m+1}\theta\,d\theta=z^{-\frac{1}{2}}J_{m+\frac{1}{2}}(z).$$

(Hobson.)

20. In the equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1+\frac{n^2}{z^2}\right)y=0$$

the quantity n is real; shew that a solution is given by

$$\cos(n\log z)-\sum_{m=1}^{\infty}\frac{(-1)^m z^{2m}\cos(u_m-n\log z)}{2^{2m}m!\,(1+n^2)^{\frac{1}{2}}(4+n^2)^{\frac{1}{2}}\ldots\ldots(m^2+n^2)^{\frac{1}{2}}},$$

where u_m denotes

$$\tan^{-1}\frac{n}{1}+\tan^{-1}\frac{n}{2}+\ldots+\tan^{-1}\frac{n}{m}.$$

(Cambridge Mathematical Tripos, Part II, 1894.)

21. Prove that the complete primitive of the differential equation

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}-\left(1+\frac{m^2}{z^2}\right)u=0,$$

where m is a positive integer, is

$$u=AI_m(z)+BK_m(z),$$

where, for real values of z,

$$I_m(z)=\frac{z^m}{1.3.5\ldots(2m-1)\,\pi}\int_0^{\pi}\cosh(z\cos\phi)\sin^{2m}\phi\,d\phi,$$

$$K_m(z)=\frac{(-1)^m z^m}{1.3.5\ldots(2m-1)}\int_0^{\infty}e^{-z\cosh\phi}\sinh^{2m}\phi\,d\phi.$$

Prove also that

$$K_m(z)=1.3.5\ldots(2m-1)(-z)^m\int_0^{\infty}(u^2+z^2)^{-m-\frac{1}{2}}\cos u\,du.$$

Shew that for very small values of z,

$$K_0(z) = -\log \frac{z}{2} - \cdot 577 \ldots,$$

and that for very large values of z,

$$K_0(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}.$$

(Cambridge Mathematical Tripos, Part II, 1898.)

22. If C be any curve in the complex domain, and m and n are integers, shew that

$$\int_C J_m(z) J_n(z)\, dz = 0,$$

$$\int_C O_m(z) O_n(z)\, dz = 0,$$

$$\int_C J_m(z) O_n(z)\, dz = k,$$

where $k=0$ if the curve does not include the origin ; and, if the curve does include the origin,

$$k = 0 \quad \text{if } m \neq n,$$

$$k = 2\pi i \text{ if } m = n.$$

CHAPTER XIII.

Applications to the Equations of Mathematical Physics.

167. *Introduction: illustration of the general method.*

The functions which have been introduced in the three preceding chapters are of very great importance in the applications of mathematics to physical investigations. Such applications are outside the province of this book; but most of them depend essentially on one underlying circumstance, namely that by means of these functions it is possible to construct series which satisfy certain partial differential equations, known as *the partial differential equations of mathematical physics*; and in this chapter it is proposed to explain and illustrate this fundamental property.

The general method may be explained by considering first the solution of the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

a solution which, while resting on the same principles as those to be developed later, does not require the use of any but the elementary functions of analysis.

Consider any solution $V(x, y)$ of this equation (1). Near any point at which a branch of the function $V(x, y)$ is a regular function of x and y, and which we may without loss of generality take as origin of coordinates, this branch of the function $V(x, y)$ can by Taylor's Theorem be expanded as a power-series of the form

$$V(x, y) = a_0 + a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + \ldots\ldots;$$

on substituting this value of V in equation (1), and equating to zero the coefficients of the various powers of x and y, we obtain the relations

$$a_2 + c_2 = 0,$$

$$3a_3 + c_3 = 0,$$

$$3d_3 + b_3 = 0,$$

$$\ldots\ldots\ldots\ldots$$

Fixing our attention on those terms in V which are homogeneous of the nth degree in x and y combined, it is clear that the equalities just written will furnish $(n-1)$ relations between the $(n+1)$ coefficients of these terms of degree n. When these equations are satisfied, there will therefore remain only $\{(n+1)-(n-1)\}$ or 2 coefficients really arbitrary in the terms of the nth degree in V.

Now the expressions

$$V=(x+iy)^n$$

and

$$V=(x-iy)^n$$

satisfy equation (1), and therefore if A_n and B_n are any arbitrary constants, the expression

$$A_n(x+iy)^n+B_n(x-iy)^n$$

satisfies equation (1), and is homogeneous of the nth degree in x and y, and contains two arbitrary constants. It therefore represents the most general form of the terms of the nth degree in V; and so the general solution of equation (1), regular at the origin, can be expressed in the form

$$V(x,y)=A_0+A_1(x+iy)+B_1(x-iy)+A_2(x+iy)^2+B_2(x-iy)^2+\dots \quad \dots(2),$$

where the quantities A_0, A_1, B_1, A_2, ... are arbitrary constants.

This expansion furnishes the *general* solution of equation (1); what is however in general needed is the *particular* solution of equation (1) which satisfies some further conditions. As an example of the conditions most frequently occurring, we shall suppose that the value of the required solution $V(x, y)$ is known at every point of the circumference of a circle, whose centre is at the origin and whose radius is any quantity a; it being supposed that this circle lies wholly within the region for which V is regular. This being given, we shall shew that the constants A_0, A_1, B_1, ... can be found, and the solution can be completely determined.

For writing

$$x=r\cos\theta, \qquad y=r\sin\theta,$$

the value of V is known when $r=a$, as a function of θ, say $f(\theta)$. Let the function $f(\theta)$ be expanded as a Fourier series in the form

$$f(\theta)=a_0+a_1\cos\theta+b_1\sin\theta+a_2\cos 2\theta+b_2\sin 2\theta+\dots \quad \dots\dots(3),$$

where the coefficients a_0, a_1, b_1, a_2, ... are given by the formulae

$$\left.\begin{aligned} a_0&=\frac{1}{2\pi}\int_0^{2\pi}f(t)\,dt \\ a_n&=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos nt\,dt \\ b_n&=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin nt\,dt \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots(4).$$

Consider now the expression

$$a_0 + \frac{r}{a}(a_1 \cos\theta + b_1 \sin\theta) + \left(\frac{r}{a}\right)^2 (a_2 \cos 2\theta + b_2 \sin 2\theta) + \dots \dots (5).$$

This expression (5) reduces to (3), i.e. to $f(\theta)$, when $r=a$; and since we have

$$r^n \cos n\theta = \frac{1}{2}\{(x+iy)^n + (x-iy)^n\},$$

$$r^n \sin n\theta = \frac{1}{2i}\{(x+iy)^n - (x-iy)^n\},$$

it is clear that the expression (5) is of the form (2), i.e. that it is a solution of the equation (1).

It follows that *the solution V of equation* (1), *which is characterised by the condition that it has the value $V=f(\theta)$ when $r=a$, is given by the expansion*

$$V = a_0 + \frac{r}{a}(a_1 \cos\theta + b_1 \sin\theta) + \left(\frac{r}{a}\right)^2 (a_2 \cos 2\theta + b_2 \sin 2\theta) + \dots,$$

where

$$\begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)\,dt, \\ a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\cos nt\,dt, \\ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\sin nt\,dt. \end{cases}$$

The principal object of this chapter will be to obtain theorems analogous to this for the other partial differential equations of mathematical physics; the method followed will be in most respects similar to that by which this result has been obtained.

168. *Laplace's equation; the general solution; certain particular solutions.*

The partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

is known as *Laplace's equation*, or the *potential-equation*, and is of importance in the investigations of mathematical physics.

The general solution of this equation was given by the author in 1902. It may be written

$$V = \int_0^{2\pi} f(x\cos t + y\sin t + iz,\, t)\,dt,$$

where f is any arbitrary function of the two arguments $x\cos t + y\sin t + iz$ and t. The solution is effected in *Monthly Notices of the Royal Astron. Soc.*, Vol. LXII. In this chapter however we are concerned not so much with the *general* solution as with the *particular* solutions which satisfy certain further conditions. To the consideration of these we shall now proceed.

Let the equation be transformed by taking instead of the independent variables x, y, z, a new set of independent variables r, θ, ϕ, connected with them by the relations

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta. \end{cases}$$

It is found without difficulty* that Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial}{\sin\theta\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

Let us seek for particular solutions of this equation, of the form

$$V = R\Theta\Phi,$$

where R, Θ, Φ, are functions respectively of r alone, θ alone, and ϕ alone.

Substituting, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0.$$

Now the quantity

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

does not involve θ or ϕ; and since by this equation it is equal to

$$-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2},$$

it clearly cannot vary with r: it is therefore independent of r, θ, and ϕ, and so must be a constant; this constant we shall write in the form $n(n+1)$.

We thus have

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0.$$

Write $r = e^u$, so $dr = e^u du$. Then this equation becomes

$$e^{-u}\frac{d}{du}\left(e^u\frac{dR}{du}\right) - n(n+1)R = 0$$

or

$$\frac{d^2R}{du^2} + \frac{dR}{du} - n(n+1)R = 0.$$

* The work is given in full in Edwards' *Differential Calculus.*

This is a linear differential equation of the second order with constant coefficients; its solution, found in the usual way, is

$$Ae^{nu} + Be^{-(n+1)u},$$

where A and B are arbitrary constants.

The most general form of the function R is therefore

$$R = Ar^n + Br^{-n-1}.$$

Considering next the function Φ, it can in the same way be shewn that the quantity

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

is independent of r, θ, and ϕ, and so must be a constant. Writing this constant in the form $-m^2$, we have for the determination of Φ the equation

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0,$$

of which the general solution is

$$\Phi = a\cos m\phi + b\sin m\phi,$$

where a and b are arbitrary constants.

It thus appears that the expressions

$$r^n \cos m\phi\,\Theta \quad \text{and} \quad r^n \sin m\phi\,\Theta$$

are particular solutions of Laplace's equation, if n and m are any constants and Θ is a function (of θ only) which satisfies the equation

$$n(n+1) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0.$$

Writing $\cos\theta = z$, this becomes

$$\frac{d}{dz}\left\{(1-z^2)\frac{d\Theta}{dz}\right\} + \left\{n(n+1) - \frac{m^2}{1-z^2}\right\}\Theta = 0.$$

But when m is a positive integer, this is (§ 129) the equation which is satisfied by the associated Legendre functions of order n and degree m; so a particular solution is the function

$$P_n^m(z), \quad \text{or} \quad P_n^m(\cos\theta).$$

Hence generally we see that *the* $(2n+1)$ *expressions*

$$r^n P_n(\cos\theta),\ r^n\cos\phi\, P_n^1(\cos\theta),\ r^n\cos 2\phi\, P_n^2(\cos\theta),\ \ldots,\ r^n\cos n\phi\, P_n^n(\cos\theta),$$
$$r^n\sin\phi\, P_n^1(\cos\theta),\ r^n\sin 2\phi\, P_n^2(\cos\theta),\ \ldots,\ r^n\sin n\phi\, P_n^n(\cos\theta),$$

where n is a positive integer, are particular solutions of Laplace's equation.

Moreover, since $P_n^m(\cos\theta)$ is of the form $\sin^m\theta \times$ a polynomial of degree $(n-m)$ in $\cos\theta$, it is easily seen that each of these quantities, if expressed in terms of x, y, z, becomes a polynomial, homogeneous of degree n, in x, y, z. It can in fact be easily shewn, by using the result of § 132, that

$$r^n \cos m\phi\, P_n^m(\cos\theta)$$

is a constant multiple of

$$\int_0^{2\pi} (x\cos t + y\sin t + iz)^n \cos mt\, dt,$$

and that

$$r^n \sin m\phi\, P_n^m(\cos\theta)$$

is a constant multiple of

$$\int_0^{2\pi} (x\cos t + y\sin t + iz)^n \sin mt\, dt,$$

from which their polynomial character is evident; these forms have the further advantage of exhibiting these particular solutions as cases of the general solution given at the beginning of this article.

Example. If coordinates r, θ, ϕ are defined by the equations

$$\begin{cases} x = r\cos\theta, \\ y = (r^2-1)^{\frac{1}{2}}\sin\theta\cos\phi, \\ z = (r^2-1)^{\frac{1}{2}}\sin\theta\sin\phi, \end{cases}$$

shew that the function

$$V = P_n^m(r)\, P_n^m(\cos\theta)\cos m\phi$$

is a solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

169. *The series-solution of Laplace's equation.*

The particular solutions of Laplace's equation, which have been found in the preceding article, enable us to express the general solution, in the form of an infinite series involving Legendre functions. This series-solution will of course be really equivalent to an expansion of the general solution

$$\int_0^{2\pi} f(x\cos t + y\sin t + iz,\, t)\, dt$$

already mentioned; but the series-form is (as will appear from § 170) more convenient in determining solutions which satisfy given boundary-conditions.

For let $V(x, y, z)$ be any solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Then in the neighbourhood of any ordinary point, which we may take as the origin of coordinates, V can be expanded in the form

$$V = a_0 + a_1 x + b_1 y + c_1 z + a_2 x^2 + b_2 y^2 + c_2 z^2 + d_2 yz + e_2 zx + f_2 xy + a_3 x^3 + \ldots.$$

Substituting this expansion in Laplace's equation, and equating to zero the coefficients of the various powers of x, y, z, we obtain an infinite number of linear relations between the coefficients a_0, a_1, b_1, c_1, a_2, ...

There are $\frac{1}{2}n(n-1)$ relations of this kind between the $\frac{1}{2}(n+1)(n+2)$ coefficients of terms of degree n in the expansion of V: and so only $\left\{\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1)\right\}$ or $(2n+1)$ of the coefficients of terms of degree n in the expansion of V are really independent. But in the last article we have found $(2n+1)$ independent polynomials of degree n in x, y, z, which satisfy Laplace's equation, namely the quantities

$$r^n P_n(\cos\theta),\quad r^n \cos\phi P_n^1(\cos\theta),\ \ldots\ldots,\ r^n \cos n\phi P_n^n(\cos\theta),$$
$$r^n \sin\phi P_n^1(\cos\theta),\ \ldots\ldots,\ r^n \sin n\phi P_n^n(\cos\theta).$$

It follows that the terms which are of degree n in x, y, z in the expansion of V must be a linear combination of these $(2n+1)$ quantities; that is, V must be expansible in the form

$$\begin{aligned} V = A_0 &+ r\{A_1 P_1(\cos\theta) + A_1^1 \cos\phi P_1^1(\cos\theta) + B_1^1 \sin\phi P_1^1(\cos\theta)\} \\ &+ r^2\{A_2 P_2(\cos\theta) + A_2^1 \cos\phi P_2^1(\cos\theta) + A_2^2 \cos 2\phi P_2^2(\cos\theta) \\ &+ B_2^1 \sin\phi P_2^1(\cos\theta) + B_2^2 \sin 2\phi P_2^2(\cos\theta)\} + \ldots, \end{aligned}$$

where the quantities A_0, A_1, A_1^1, B_1^1, ... are arbitrary constants.

170. *Determination of a solution of Laplace's equation which satisfies given boundary conditions.*

In order to determine the unknown constants A_0, A_1, A_1^1, B_1^1, ..., which appear in the expansion just found, it is necessary to know the remaining conditions which the function V is required to satisfy. A condition of frequent occurrence is that V is to have certain assigned values at the points of the surface of a sphere, which we may take as being of radius a and having its centre at the origin. This sphere will be supposed to lie entirely within the region for which V is a regular function of its arguments x, y, z. When $r = a$, V is therefore to be equal to a given function $f(\theta, \phi)$ of θ and ϕ. The constants A_0, A_1, A_1^1, B_1^1, ..., are therefore to be determined from the equation

$$\begin{aligned} f(\theta, \phi) = A_0 &+ a\{A_1 P_1(\cos\theta) + A_1^1 \cos\phi\, P_1^1(\cos\theta) + B_1^1 \sin\phi P_1^1(\cos\theta)\} \\ &+ a^2\{A_2 P_2(\cos\theta) + A_2^1 \cos\phi P_2^1(\cos\theta) + \ldots\} + \ldots \end{aligned}$$

In order to obtain the value of one of these constants, say A_n^m, from this equation, we multiply both sides of the equation by $P_n^m(\cos\theta)\cos m\phi$, and integrate over the surface of the sphere. On the left-hand side we thus have

$$\int_0^\pi \int_0^{2\pi} f(\theta, \phi) P_n^m(\cos\theta)\cos m\phi \sin\theta \, d\theta \, d\phi.$$

As to the right-hand side, we know that

$$\int_0^{2\pi} \cos m\phi \cos r\phi \, d\phi$$

is zero except when $r = m$, and that

$$\int_0^{2\pi} \cos m\phi \sin r\phi \, d\phi$$

is always zero; and also (by § 130) that

$$\int_0^\pi P_r^m(\cos\theta) P_n^m(\cos\theta)\sin\theta \, d\theta$$

is zero except when $r = n$. It follows that on the right-hand side, every term vanishes except the term

$$a^n A_n^m \int_0^\pi \int_0^{2\pi} \{P_n^m(\cos\theta)\}^2 \cos^2 m\phi \sin\theta \, d\theta \, d\phi.$$

Since $$\int_0^{2\pi} \cos^2 m\phi \, d\phi = \pi,$$

and (by § 130) $$\int_0^\pi \{P_n^m(\cos\theta)\}^2 \sin\theta \, d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!},$$

this term has the value

$$\pi a^n A_n^m \cdot \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}.$$

We have therefore the formula

$$A_n^m = \frac{2n+1}{2\pi a^n} \cdot \frac{(n-m)!}{(n+m)!}\int_0^\pi \int_0^{2\pi} f(\theta, \phi) P_n^m(\cos\theta)\cos m\phi \sin\theta \, d\theta \, d\phi,$$

which determines the coefficients A_n^m in the expansion of V.

The coefficients B_n^m can be similarly determined: and so finally *the solution V of Laplace's equation, which has the value $f(\theta, \phi)$ at the surface of the sphere, is given for points in the interior of the sphere by the expansion*

$$V = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\left(\frac{r}{a}\right)^n \int_0^\pi \int_0^{2\pi} f(\theta', \phi') \Big\{P_n(\cos\theta') P_n(\cos\theta)$$

$$+ 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi - \phi')\Big\} \sin\theta' d\theta' d\phi'.$$

This result may be regarded as a three-dimensional analogue of the two-dimensional result of § 167.

Example 1. Shew, by applying the expansion-theorem just given, that

$$P_n \{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\} = P_n(\cos\theta)\,P_n(\cos\theta')$$

$$+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta')\cos m(\phi-\phi').$$

Example 2. Prove that if the product of a homogeneous polynomial of degree n in x, y, z and the function $P_n\{\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\}$ be integrated over the surface of the sphere, the result is $4\pi/(2n+1)$ multiplied by the value of the polynomial at the point (θ', ϕ').

(This can be proved by taking θ' to be zero, which involves no real loss of generality, and expanding the polynomial by the theorem of this article.)

171. *Particular solutions of Laplace's equation which depend on Bessel functions.*

It is possible to construct solutions of Laplace's equations in series in several ways, of which that which has been given, and which depends on Legendre functions, may be taken as representative. A full discussion of the other methods would be beyond the scope of this book, but a general idea of them may be inferred from the result which will next be established, namely that the Bessel functions furnish a group of particular solutions of Laplace's equation, just as the Legendre functions do.

When Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is expressed in terms of the "cylindrical coordinates" z, ρ, ϕ, where ρ and ϕ are defined by the equations

$$\begin{cases} x=\rho\cos\phi, \\ y=\rho\sin\phi, \end{cases}$$

it takes the form

$$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}=0.$$

Let us seek for particular solutions of this equation, of the form

$$V = Z\mathrm{P}\Phi,$$

where Z, P, Φ, are functions of z alone, ρ alone, and ϕ alone, respectively.

On substituting this value of V, Laplace's equation becomes

$$\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{\mathrm{P}}\left(\frac{d^2\mathrm{P}}{d\rho^2}+\frac{1}{\rho}\frac{d\mathrm{P}}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2}=0.$$

This equation shews that the quantity

$$\frac{1}{Z}\frac{d^2Z}{dz^2}$$

must be a constant independent of z, ρ, and ϕ; let this constant be denoted by k^2. Then on solving the equation

$$\frac{d^2Z}{dz^2} = k^2 Z,$$

we have the particular solutions

$$Z = e^{kz} \text{ and } Z = e^{-kz}.$$

Similarly the quantity

$$\frac{1}{\phi}\frac{d^2\Phi}{d\phi^2}$$

is a constant, which may be denoted by $-m^2$; on solving the equation

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0,$$

we obtain the particular solutions

$$\Phi = \cos m\phi \text{ and } \Phi = \sin m\phi.$$

The equation to determine P is now

$$\frac{d^2\mathrm{P}}{d\rho^2} + \frac{1}{\rho}\frac{d\mathrm{P}}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)\mathrm{P} = 0.$$

On putting $k\rho = y$, this becomes Bessel's equation of order m,

$$\frac{d^2\mathrm{P}}{dy^2} + \frac{1}{y}\frac{d\mathrm{P}}{dy} + \left(1 - \frac{m^2}{y^2}\right)\mathrm{P} = 0,$$

a particular solution of which is

$$\mathrm{P} = J_m(y).$$

It follows that *the expressions*

$$e^{\pm kz}\cos m\phi J_m(k\rho) \text{ and } e^{\pm kz}\sin m\phi J_m(k\rho),$$

where k and m are arbitrary constants, are particular solutions of Laplace's equation.

172. *Solution of the equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0.$$

We now proceed to consider another partial differential equation.

We have seen in the last article that Laplace's equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=0$$

is satisfied by the particular solutions

$$e^z J_n(r)\cos n\theta \text{ and } e^z J_n(r)\sin n\theta,$$

where $$x=r\cos\theta,\ y=r\sin\theta.$$

But if we write $$W=e^z V,$$

where V is a function of x and y only, the Laplace's equation for W becomes

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0.$$

It follows that, for all values of n, *the quantities*

$$J_n(r)\cos n\theta \text{ and } J_n(r)\sin n\theta$$

are particular solutions of this latter equation.

From these particular solutions, as in the case of the solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

already described, we can build up the general solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0$$

in the form $$V=\sum_{n=0}^{\infty} J_n(r)(a_n\cos n\theta+b_n\sin n\theta),$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, are arbitrary constants.

173. *Solution of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0.$$

In order to solve the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0,$$

which is likewise of great importance in the investigations of mathematical physics, we first express the equation in terms of new independent variables r, θ, ϕ, defined by the equations

$$\begin{cases} x=r\sin\theta\cos\phi, \\ y=r\sin\theta\sin\phi, \\ z=r\cos\theta, \end{cases}$$

and then endeavour to find particular solutions of the form

$$V = R\Theta\Phi,$$

where R, Θ, Φ, are functions respectively of r alone, θ alone, and ϕ alone. Proceeding as in § 168, the differential equation becomes

$$r^2 + \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0.$$

This equation can be solved by the process used in § 168 for finding particular solutions of Laplace's equation; the quantity

$$r^2 + \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

must be a constant, which we shall denote by $n(n+1)$. If in the resulting equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \{r^2 - n(n+1)\}R = 0,$$

we write $y = Rr^{\frac{1}{2}}$, it becomes

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left\{1 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right\}y = 0,$$

which is Bessel's equation of order $\left(n+\frac{1}{2}\right)$.

The quantity R can therefore be taken to be

$$R = r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r).$$

The equations for Θ and Φ are now found to be the same as those which occur (§ 168) in the solution of Laplace's equation; and proceeding as in § 169, we find that *the general solution of the partial differential equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0,$$

regular near the origin, can be expressed in the form

$$V = \sum_{n=0}^{\infty} r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r) \times \left\{\begin{array}{l} A_nP_n(\cos\theta) + A_n^1\cos\phi P_n^1(\cos\theta) + \ldots + A_n^n\cos n\phi P_n^n(\cos\theta) \\ \qquad + B_n^1\sin\phi P_n^1(\cos\theta) + \ldots + B_n^n\sin n\phi P_n^n(\cos\theta)\end{array}\right\},$$

where the quantities A and B are arbitrary constants.

When a particular solution V of the equation is to be determined by the condition that it is to take prescribed values at all points on the surface of a sphere, the constants A and B are determined exactly as in § 170.

Example. Shew, as a case of the general expansion of this article, that

$$e^{ir\cos\theta} = \sum_{n=0}^{\infty} i^n (2\pi)^{\frac{1}{2}} (2n+1) r^{-\frac{1}{2}} P_n(\cos\theta) J_{n+\frac{1}{2}}(r).$$

Note. The partial differential equations of §§ 172, 173, possess general solutions analogous to that of Laplace's equation. The solution of the equation of § 172 is

$$V = \int_0^{2\pi} e^{i(x\cos t + y\sin t)} f(t)\, dt,$$

where f is an arbitrary function; and the solution of the equation of § 173 is

$$V = \int_0^{2\pi}\int_0^{2\pi} e^{i(x\sin t\cos\psi + y\sin t\sin\psi + z\cos t)} f(t, \psi)\, dt d\psi,$$

where f is an arbitrary function. For the proof of these results, reference may be made to papers by the author.

Miscellaneous Examples.

1. If a solution V of Laplace's equation be symmetrical with respect to the axis of z, and have the value $V = f(z)$ at points on that axis, shew that its value at any other point of space is

$$V = \frac{1}{\pi}\int_0^{\pi} f\{z + i(x^2+y^2)^{\frac{1}{2}}\cos\phi\}\, d\phi.$$

2. Deduce from the result of Example 1 that the potential of a circular ring of mass M, whose equation is

$$x^2+y^2=c^2, \quad z=0,$$

is

$$\frac{1}{\pi}\int_0^{\pi} M[c^2 + \{z + i(x^2+y^2)^{\frac{1}{2}}\cos\phi\}^2]^{-\frac{1}{2}}\, d\phi.$$

3. Let $P(x, y, z)$ be a point in space, and let the plane through P and the axis of z make an angle ϕ with the plane zx. Let this plane cut the circle whose equations are

$$z=0, \quad x^2+y^2=k^2,$$

in the points a and γ, and let the angle $a\hat{P}\gamma$ be denoted by θ and $\log(Pa/P\gamma)$ by σ.

If σ, θ, ϕ be regarded as coordinates defining the position of the point P, shew that Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

takes the form

$$\frac{\partial}{\partial\sigma}\left\{\frac{\sinh\sigma}{\cosh\sigma - \cos\theta}\frac{\partial V}{\partial\sigma}\right\} + \frac{\partial}{\partial\theta}\left\{\frac{\sinh\sigma}{\cosh\sigma - \cos\theta}\frac{\partial V}{\partial\theta}\right\} + \frac{1}{\sinh^2\sigma(\cosh\sigma - \cos\theta)}\frac{\partial^2 V}{\partial\phi^2} = 0,$$

and that the quantities

$$V = (\cosh\sigma - \cos\theta)^{\frac{1}{2}}\cos n\theta\cos m\phi\, P^m_{n-\frac{1}{2}}(\cosh\sigma)$$

are solutions of it.

CHAPTER XIV.

THE ELLIPTIC FUNCTION $\wp(z)$.

174. *Introduction.*

If $f(z)$ denote any one of the circular functions $\sin z$, $\cos z$, $\tan z$ ..., it is well known that

$$f(z+2\pi)=f(z),$$

and hence that

$$f(z+2n\pi)=f(z),$$

where n is any positive or negative integer.

This fact is generally expressed by the statement that *the circular functions admit the period* 2π. They are on this account said to be *periodic* functions; and in contradistinction to other classes of periodic functions, which will be introduced subsequently, they are called *singly-periodic* functions.

It will in fact be established in this chapter that a class of functions exists possessing the following properties: if $f(z)$ be any function of the class, then $f(z)$ is a one-valued function of z, with no singularities other than poles in the finite part of the z-plane; moreover, $f(z)$ satisfies, for all values of z, the equations

$$f(z+2\omega_1)=f(z),$$

$$f(z+2\omega_2)=f(z),$$

where ω_1 and ω_2 are two quantities independent of z. Functions $f(z)$ of this class are said to *admit* the quantities $2\omega_1$ and $2\omega_2$ as *periods*, and are called *doubly-periodic functions* or *elliptic functions*. The two periods $2\omega_1$ and $2\omega_2$ play the same part in the theory of elliptic functions as is played by the single period 2π in the theory of circular functions.

By repeated application of the formulae written above, we obtain as the characteristic equation of all elliptic functions the equation

$$f(z+2m\omega_1+2n\omega_2)=f(z),$$

where m and n are any integers.

175. *Definition of* $\wp(z)$.

The elliptic functions may, as we have just seen, be regarded as a generalisation of the circular functions. It is natural therefore to introduce them into analysis by some definition analogous to one of the definitions used in the theory of circular functions.

One mode of developing the theory of the circular functions is to start from the infinite series

$$\frac{1}{z^2} + \sum_{m=\pm 1}^{\pm\infty} \frac{1}{(z - m\pi)^2}.$$

It can be shewn that this series converges absolutely and uniformly for all values of z except the values

$$z = 0, \quad \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi \ldots;$$

and that it admits the period 2π. If now its sum be denoted by $(\sin z)^{-2}$, and this be regarded as the definition of the function $\sin z$, then from this definition we can derive all the properties of the function $\sin z$, and thus a complete theory of the circular functions can be developed.

Similarly, as the basis of the theory of elliptic functions, we form the infinite series

$$z^{-2} + \Sigma \{(z - 2m\omega_1 - 2n\omega_2)^{-2} - (2m\omega_1 + 2n\omega_2)^{-2}\},$$

where ω_1 and ω_2 are any two quantities, independent of z, whose ratio is not purely real, and where the summation extends over all integer and zero (except simultaneous zero) values of m and of n.

It has been shewn in § 11 that this series is absolutely convergent for all values of z, except the values $z = 0, \pm\omega_1, \pm\omega_2, \pm\omega_1 \pm \omega_2, \pm 2\omega_1 \pm \omega_2, \ldots$.

By comparing the series with the convergent series $\Sigma(m^2 + n^2)^{-\frac{3}{2}}$ as in § 11, it is seen that this convergence is also uniform (§ 52). The series therefore represents a one-valued function of z, regular for all values of the variable z except the values $z = 2m\omega_1 + 2n\omega_2$; and at these points, which are the singularities of the function, it clearly has poles of the second order.

We shall denote this function by the symbol $\wp(z)$. Its introduction is due to Weierstrass.

There are other ways of introducing both the circular and elliptic functions into Analysis; for the circular functions, the following may be mentioned:

(1) The geometrical definition, according to which $\sin z$ is the ratio of one side to the hypotenuse, in a right-angled triangle of which one angle is z. This is the definition usually given in the introductory chapter of treatises on Trigonometry: but from our point of view it is defective, as it applies only to real values of z.

(2) The definition by means of the infinite product

$$\sin z = z\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\left(1 - \frac{z^2}{3^2\pi^2}\right)\ldots.$$

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(3) The definition by the inversion of a definite integral,

$$z = \int_0^{\sin^{-1} z} (1 - t^2)^{-\frac{1}{2}} \, dt.$$

We shall see subsequently that alternative definitions of the elliptic functions exist, analogous to each of these definitions (1), (2), (3), and that they may if desired be taken as fundamental in the theory.

Example. Prove that

$$\wp(z) = C + \left(\frac{\pi}{2\omega_1}\right)^2 \sum_{n=-\infty}^{\infty} \operatorname{cosec}^2 \left(\frac{z - 2n\omega_2}{2\omega_1}\pi\right),$$

where

$$C = -\left(\frac{\pi}{2\omega_1}\right)^2 \left\{\frac{1}{3} + \sum_{n=-\infty}^{\infty} \operatorname{cosec}^2 \frac{2n\omega_2}{\omega_1}\pi\right\}.$$

176. *Periodicity, and other properties, of $\wp(z)$.*

The function $\wp(z)$ is an *even* function of z, i.e. it satisfies the equation

$$\wp(z) = \wp(-z).$$

For if $-z$ be substituted for z in the series which defines $\wp(z)$, the resulting series is the same as the original series, except that the order of the terms is changed. But since the series is absolutely convergent, this change in order does not affect the value of the sum of the series; and therefore we have

$$\wp(z) = \wp(-z).$$

Further, the function $\wp(z)$ admits the quantity $2\omega_1$ as a period.

For

$$\begin{aligned} \wp(z + 2\omega_1) - \wp(z) \\ &= (z + 2\omega_1)^{-2} - z^{-2} + \Sigma \{(z + 2\omega_1 - 2m\omega_1 - 2n\omega_2)^{-2} - (z - 2m\omega_1 - 2n\omega_2)^{-2}\} \\ &= \Sigma \{(z - 2(m-1)\omega_1 - 2n\omega_2)^{-2} - (z - 2m\omega_1 - 2n\omega_2)^{-2}\}, \end{aligned}$$

where the last summation is extended over all integer and zero values of m and n without exception. But this last sum is zero, since its terms destroy each other in pairs. Thus we have

$$\wp(z + 2\omega_1) = \wp(z).$$

Similarly $$\wp(z + 2\omega_2) = \wp(z),$$

and generally $$\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z),$$

where m and n are any integers.

Therefore *the function $\wp(z)$ admits the two periods $2\omega_1$ and $2\omega_2$.*

Differentiating the above results, we see that $\wp'(z)$ *is an odd function of z, and admits the same periods as $\wp(z)$.*

177. *The period-parallelograms.*

The study of elliptic functions is much facilitated by a method of geometrical representation which will now be explained.

Suppose that in the plane of the variable z we mark the points $z=0$, $z=2\omega_1$, $z=2\omega_2$, $z=2\omega_1+2\omega_2$, ... and generally all the points comprised in the formula $z=2m\omega_1+2n\omega_2$, where m and n are any positive or negative integers or zero.

By joining the point $z=0$ by a straight line to the point $z=2\omega_1$, then joining the point $2\omega_1$ to the point $2\omega_1+2\omega_2$, then joining the point $2\omega_1+2\omega_2$ to the point $2\omega_2$, and lastly joining the point $2\omega_2$ to the point $z=0$, we obtain a parallelogram in the z-plane, which we shall call the *fundamental period-parallelogram.*

It is clear that the whole z-plane may be covered with a network of parallelograms, which are each similar and equal to this parallelogram, and which can be obtained by joining the other marked points by straight lines. These parallelograms will be called *period-parallelograms.*

Then if t be any quantity, the points

$$z=t,\ z=t+2\omega_1,\ z=t+2\omega_2,\ \ldots,\ z=t+2m\omega_1+2n\omega_2,$$

manifestly occupy corresponding positions in these parallelograms; these points are said to be *congruent* to each other.

It follows from the fundamental property of $\wp(z)$ that *the function* $\wp(z)$ *has the same value at all points which are congruent with each other*; and hence that *the values which the function* $\wp(z)$ *has in any period-parallelogram are a mere repetition of the values which the function has in any other period-parallelogram.*

178. *Expression of the function* $\wp(z)$ *by means of an integral.*

We shall now obtain a form for $\wp(z)$ in terms of an integral, which will be found to be of great importance in the theory of the function.

The quantity $$\wp(z)-z^{-2},$$
or $$\Sigma\,\{(z-2m\omega_1-2n\omega_2)^{-2}-(2m\omega_1+2n\omega_2)^{-2}\},$$
is a regular function of z in the neighbourhood of the point $z=0$, and is an even function of z. It can therefore by Taylor's theorem be expanded, for points z near the origin, in the form

$$\wp(z)-z^{-2}=\frac{g_2}{20}z^2+\frac{g_3}{28}z^4+\ldots,$$

where clearly we shall have

$$\frac{g_2}{20}=3\Sigma\,(2m\omega_1+2n\omega_2)^{-4},$$

$$\frac{g_3}{28}=5\Sigma\,(2m\omega_1+2n\omega_2)^{-6}.$$

Thus $$\wp(z)=z^{-2}+\frac{g_2}{20}z^2+\frac{g_3}{28}z^4+\ldots.$$

Forming the square and the derivates of this expansion, we have

$$\wp^2(z) = z^{-4} + \frac{g_2}{10} + \frac{g_3}{14} z^2 + \ldots,$$

$$\wp'(z) = -2z^{-3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \ldots,$$

$$\wp''(z) = 6z^{-4} + \frac{g_2}{10} + \frac{3}{7} g_3 z^2 + \ldots.$$

Therefore $\wp^2(z) - \frac{1}{6}\wp''(z) = \frac{1}{12} g_2 +$ terms involving z^4 at least.

It follows that the function

$$\wp^2(z) - \frac{1}{6}\wp''(z)$$

is regular in the neighbourhood of the point $z = 0$; and as it is doubly-periodic (for clearly any power or derivate of an elliptic function is likewise an elliptic function) it must be regular in the neighbourhood of each of the points

$$z = 2m\omega_1 + 2n\omega_2.$$

But the only singularities of $\wp(z)$ are at these points: and therefore the only possible singularities of the function

$$\wp^2(z) - \frac{1}{6}\wp''(z)$$

are at these points. The latter function is consequently regular for all values of z; and so by Liouville's theorem (§ 47) is independent of z, and therefore is equal to the value which it has at the point $z = 0$, which is $\frac{1}{12} g_2$.

We have therefore the relation

$$\wp^2(z) = \frac{1}{6}\wp''(z) + \frac{1}{12} g_2.$$

Multiplying by $3\wp'(z)$ and integrating, we have

$$\wp^3(z) = \frac{1}{4}\wp'^2(z) + \frac{1}{4} g_2 \wp(z) + c,$$

where c is a constant; on substituting the expansions in this equality, we find that $c = \frac{1}{4} g_3$.

Thus, finally, the function $\wp(z)$ satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3,$$

where g_2 and g_3 (called the *invariants*) are given in terms of the periods of $\wp(z)$ by the equations

$$g_2 = 60\Sigma(2m\omega_1 + 2n\omega_2)^{-4},$$

$$g_3 = 140\Sigma(2m\omega_1 + 2n\omega_2)^{-6}.$$

This differential equation can be written in the form

$$\left(\frac{dt}{dz}\right)^2 = 4t^3 - g_2 t - g_3,$$

where $$t = \wp(z),$$

and therefore (since $\wp(z)$ is infinite when z is zero) we have

$$z = \int_{\wp(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt,$$

which is the required expression of $\wp(z)$ in terms of an integral.

The preceding theorems may be illustrated by the results which correspond to them in the theory of the circular functions. Thus we may in the following way discuss the properties of a function $f(z)$ (really *cosec*$^2 z$), which we shall take to be defined by the series

$$f(z) = z^{-2} + (z-\pi)^{-2} + (z+\pi)^{-2} + (z-2\pi)^{-2} + (z+2\pi)^{-2} + (z-3\pi)^{-2} + \ldots.$$

This series is clearly infinite at the points $z=0, \pi, -\pi, 2\pi, \ldots$; for other values of z it is absolutely and uniformly convergent, as is seen by comparing it with the series

$$1 + 1^{-2} + 1^{-2} + 2^{-2} + 2^{-2} + 3^{-2} + 3^{-2} + \ldots.$$

The effect of adding any multiple of π to z is to produce a new series whose terms are the terms of the original series, arranged in a different order; this does not affect the sum of the series, since the convergence is absolute; and therefore $f(z)$ *is a periodic function of* z, *with the period* π.

By drawing parallel lines in the z-plane at distances π from each other, we therefore divide the plane into strips, such that at points occupying corresponding positions in the different strips, $f(z)$ has the same value. In each strip, $f(z)$ has only one singularity, namely at that one of the points $0, \pi, -\pi, 2\pi, -2\pi, \ldots$ which lies within the strip. The function is not infinite at the infinite ends of the strip, because the several terms of the series for $f(z)$ are then small compared with the corresponding terms of the comparison-series

$$1 + 1^{-2} + 1^{-2} + 2^{-2} + 2^{-2} + 3^{-2} + 3^{-2} + \ldots.$$

Now near the point $z=0$, the function $f(z)$ can be written in the form

$$\begin{aligned} f(z) &= z^{-2} + \pi^{-2}\left(1 - \frac{z}{\pi}\right)^{-2} + \pi^{-2}\left(1 + \frac{z}{\pi}\right)^{-2} + (2\pi)^{-2}\left(1 - \frac{z}{2\pi}\right)^{-2} + \ldots \\ &= z^{-2} + \pi^{-2}(1 + 1 + 2^{-2} + 2^{-2} + \ldots) + \pi^{-4}z^2(3 + 3 + 3 \cdot 2^{-4} + 3 \cdot 2^{-4} + \ldots) + \ldots \\ &= z^{-2} + 2\pi^{-2} \cdot \frac{\pi^2}{6} + \pi^{-4}z^2 \cdot 3 \cdot 2 \cdot \frac{\pi^4}{90} + \ldots \\ &= z^{-2} + \frac{1}{3} + \frac{1}{15} z^2 + \ldots \end{aligned}$$

Differentiating and squaring this equation, we have

$$f''(z) = 6z^{-4} + \frac{2}{15} + \ldots,$$

$$f^2(z) = z^{-4} + \frac{2}{3}z^{-2} + \frac{2}{15} + \frac{1}{9} + \ldots.$$

It follows that

$$f''(z) - 6f^2(z) + 4f(z)$$

is a series containing no negative powers of z; it has therefore no singularity at the point $z=0$, and therefore (since that is the only possible singularity) no singularity in the strip which contains $z=0$, and therefore (on account of the periodic property) no singularity in any strip. It is therefore, by Liouville's theorem (§ 47), a constant: this constant must be equal to the value of the function at the point $z=0$, which (on substituting the expansions) is found to be zero. We have therefore

$$f''(z)-6f^2(z)+4f(z)=0.$$

Multiplying by $2f'(z)$ and integrating, we have

$$f'^2(z)=4f^3(z)-4f^2(z)+c,$$

where c is a constant. On substituting the expansions, c is found to be zero, and therefore

$$f'^2(z)=4f^2(z)\{f(z)-1\}$$

or

$$\left(\frac{dt}{dz}\right)^2=4t^2(t-1), \text{ where } t=f(z),$$

which gives

$$2z=\int_{f(z)}^{\infty} t^{-1}(t-1)^{-\frac{1}{2}}\,dt$$

as the expression of $f(z)$ by means of an integral.

Example. If $y=\wp(z)$, shew that

$$-\frac{1}{2}\frac{\dfrac{d^3y}{dz^3}}{\left(\dfrac{dy}{dz}\right)^3}+\frac{3}{4}\frac{\left(\dfrac{d^2y}{dz^2}\right)^2}{\left(\dfrac{dy}{dz}\right)^4}=\frac{3}{16}\{(y-e_1)^{-2}+(y-e_2)^{-2}+(y-e_3)^{-2}\}-\frac{3}{8}y\,(y-e_1)^{-1}(y-e_2)^{-1}(y-e_3)^{-1},$$

where e_1, e_2, e_3 are the roots of the equation

$$4y^3-g_2y-g_3=0.$$

For we have

$$\wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3,$$

and so

$$\left(\frac{dy}{dz}\right)^2=4(y-e_1)(y-e_2)(y-e_3).$$

Differentiating logarithmically, we have

$$\frac{2\dfrac{d^2y}{dz^2}}{\left(\dfrac{dy}{dz}\right)^2}=(y-e_1)^{-1}+(y-e_2)^{-1}+(y-e_3)^{-1}.$$

Differentiating again, we have

$$\frac{2\dfrac{d^3y}{dz^3}}{\left(\dfrac{dy}{dz}\right)^3}-\frac{4\left(\dfrac{d^2y}{dz^2}\right)^2}{\left(\dfrac{dy}{dz}\right)^4}=-(y-e_1)^{-2}-(y-e_2)^{-2}-(y-e_3)^{-2}.$$

Adding the last equation, multiplied by $\frac{1}{4}$, to the square of the preceding equation, multiplied by $\frac{3}{16}$, we have the required result.

It may be noted that the left-hand side of the equation is half the Schwartzian derivative of z with respect to y; and hence the result shews that z is the quotient of two solutions of the equation

$$\frac{d^2v}{dy^2}+\left\{\frac{3}{16}\sum_{r=1}^{3}(y-e_r)^{-2}-\frac{3}{8}y\,(y-e_1)^{-1}(y-e_2)^{-1}(y-e_3)^{-1}\right\}v=0.$$

179. *The homogeneity of the function* $\wp(z)$.

When the Weierstrassian elliptic function is considered as depending on its arguments and periods, it has a certain property of homogeneity, which will now be investigated.

Let $\wp\left(z, \begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)$ denote the function formed with the argument z and periods $2\omega_1$ and $2\omega_2$. Then we have

$$\wp\left(\lambda z, \begin{matrix}\lambda\omega_1\\ \lambda\omega_2\end{matrix}\right) = \lambda^{-2}z^{-2} + \Sigma\left\{(\lambda z - 2m\lambda\omega_1 - 2n\lambda\omega_2)^{-2} - (2m\lambda\omega_1 + 2n\lambda\omega_2)^{-2}\right\}$$

$$= \lambda^{-2}\wp\left(z, \begin{matrix}\omega_1\\ \omega_2\end{matrix}\right).$$

It follows that *the effect of multiplying the argument and the periods by the same quantity λ is equivalent to multiplying the function by λ^{-2}.*

This relation can also be expressed in terms of the quantities g_2, g_3.

For let $\wp(z;\ g_2;\ g_3)$ denote the function formed with the invariants g_2 and g_3. Then we have

$$g_2 = 60\,\Sigma\,(2m\omega_1 + 2n\omega_2)^{-4},$$

$$g_3 = 140\,\Sigma\,(2m\omega_1 + 2n\omega_2)^{-6}.$$

The effect of replacing ω_1 and ω_2 by $\lambda\omega_1$ and $\lambda\omega_2$ respectively is therefore to replace g_2 and g_3 by $\lambda^{-4}g_2$ and $\lambda^{-6}g_3$ respectively; and thus we have

$$\wp(z;\ g_2, g_3) = \wp\left(z, \begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)$$

$$= \lambda^2\wp\left(\lambda z, \begin{matrix}\lambda\omega_1\\ \lambda\omega_2\end{matrix}\right)$$

$$= \lambda^2\wp(\lambda z;\ \lambda^{-4}g_2,\ \lambda^{-6}g_3),$$

which expresses the homogeneity-property in terms of the invariants.

Example. Deduce the last result directly from the equation

$$z = \int_{\wp(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}}\,dt.$$

180. *The addition-theorem for the function* $\wp(z)$.

The function $\wp(z)$ possesses an *addition-theorem*, i.e. a formula which gives the value of $\wp(z+y)$ in terms of the values of $\wp(z)$ and $\wp(y)$, where z and y are any quantities.

To obtain this formula, consider the expression

$$\begin{vmatrix} 1 & \wp(z+y) & -\wp'(z+y) \\ 1 & \wp(z) & \wp'(z) \\ 1 & \wp(y) & \wp'(y) \end{vmatrix}$$

as a function of z.

Since it is compounded of doubly-periodic functions, it is itself a doubly-periodic function; and the only points at which it can have singularities are the points at which the functions $\wp(z+y)$ and $\wp(z)$ have singularities, i.e. the points $z=0$, $z=-y$, and points congruent (§ 177) with these.

Now for points z near the point $z=0$, we can write the determinant in the form

$$\begin{vmatrix} 1 & \wp(y)+z\wp'(y)+\frac{1}{2}z^2\wp''(y)+\dots & -\wp'(y)-z\wp''(y)-\dots \\ 1 & z^{-2}+\frac{1}{20}g_2z^2+\dots & -2z^{-3}+\frac{1}{10}g_2z+\dots \\ 1 & \wp(y) & \wp'(y) \end{vmatrix}.$$

Expanding this determinant, we find that the terms involving negative powers of z destroy each other; the determinant can therefore, in the neighbourhood of the point $z=0$, be expanded as a series of positive powers of z; that is, the function represented by the determinant has no singularity at the point $z=0$; and therefore (by the periodic property) it has no singularity at any of the points congruent with $z=0$.

Considering next the neighbourhood of the point $z=-y$, write $z=-y+x$. The determinant can be written in the form

$$\begin{vmatrix} 1 & x^{-2}+\frac{1}{20}g_2x+\dots & -2x^{-3}+\frac{1}{10}g_2x+\dots \\ 1 & \wp(-y)+x\wp'(-y)+\dots & \wp'(-y)+x\wp''(-y)+\dots \\ 1 & \wp(y) & \wp'(y) \end{vmatrix},$$

and on expansion this is found to contain no negative powers of x. The function represented by the determinant has therefore no singularity at the point $z=-y$ or any of the congruent points.

The function has therefore no singularities, and so by Liouville's theorem (§ 47) is independent of z. But it vanishes when z has the value y, since two rows of the determinant are then identical. The determinant is therefore always zero.

We thus have the formula

$$\begin{vmatrix} 1 & \wp(z+y) & -\wp'(z+y) \\ 1 & \wp(z) & \wp'(z) \\ 1 & \wp(y) & \wp'(y) \end{vmatrix} = 0,$$

true for all values of z and y. Since, by § 178, $\wp'(z+y)$, $\wp'(z)$, $\wp'(y)$ are at once expressible in terms of $\wp(z+y)$, $\wp(z)$, $\wp(y)$, respectively, this result really expresses $\wp(z+y)$ in terms of $\wp(z)$ and $\wp(y)$. It is therefore an *addition-theorem*.

The addition-theorem may also be obtained in the following way.

Take rectangular axes Ox, Ou, in a plane; and consider the intersections of the cubic curve

$$u^2 = 4x^3 - g_2 x - g_3$$

with a straight line

$$u = mx + n.$$

The abscissae x_1, x_2, x_3 of the points of intersection are the roots of the equation $\phi(x) = 0$, where

$$\phi(x) = (mx+n)^2 - 4x^3 + g_2 x + g_3.$$

The variation δx_r in one of these abscissae, consequent on small changes δm and δn in m and n, is therefore given by the equation

$$\phi'(x_r)\,\delta x_r + 2\,(mx_r + n)\,(x_r \delta m + \delta n) = 0,$$

whence

$$\sum_{r=1}^{3} \frac{\delta x_r}{mx_r + n} = -2 \sum_{r=1}^{3} \frac{x_r \delta m + \delta n}{\phi'(x_r)}$$

$$= 0, \text{ by a well-known theorem in partial fractions.}$$

Therefore
$$\sum_{r=1}^{3} (4x_r^3 - g_2 x_r - g_3)^{-\frac{1}{2}}\, \delta x_r = 0.$$

Now when n is infinite, the abscissae x_1, x_2, x_3 are all infinite: we may therefore integrate the last equation over the series of positions of the straight line $y = mx + n$, and obtain the result

$$\sum_{r=1}^{3} \int_{x_r}^{\infty} (4x_r^3 - g_2 x_r - g_3)^{-\frac{1}{2}}\, dx_r = 0.$$

If we write

$$x_1 = \wp(z), \quad x_2 = \wp(y), \quad x_3 = \wp(w),$$

we have therefore
$$z + y + w = 0.$$

But the ordinates of the three points of intersection are

$$u_1 = \wp'(z), \quad u_2 = \wp'(y), \quad u_3 = \wp'(w).$$

Since the three points are collinear, we have

$$\begin{vmatrix} 1 & x_3 & u_3 \\ 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \end{vmatrix} = 0,$$

and therefore
$$\begin{vmatrix} 1 & \wp(z+y) & -\wp'(z+y) \\ 1 & \wp(z) & \wp'(z) \\ 1 & \wp(y) & \wp'(y) \end{vmatrix} = 0,$$

which is the addition-theorem.

181. *Another form of the addition-theorem.*

The determinantal form of the addition-theorem given in the last article may be replaced in the following way by a simpler, though less symmetrical, formula.

Consider the equation

$$\begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(z) & \wp'(z) \\ 1 & \wp(y) & \wp'(y) \end{vmatrix} = 0.$$

If in this we replace $\wp'(x)$ by its value in terms of $\wp(x)$, and expand, we have

$$\{4\wp^3(x) - g_2\wp(x) - g_3\}\{\wp(z) - \wp(y)\}^2$$
$$= [\wp'(z)\{\wp(x) - \wp(y)\} + \wp'(y)\{\wp(z) - \wp(x)\}]^2.$$

This may be regarded as a cubic equation in the quantity $\wp(x)$. One of its roots is $\wp(x) = \wp(z+y)$, by the addition-theorem; and the other two roots are $\wp(x) = \wp(z)$ and $\wp(x) = \wp(y)$, since the determinant vanishes when z or y is substituted for x. We have therefore

$$\wp(z) + \wp(y) + \wp(z+y) = \text{Sum of roots of cubic}$$
$$= -\{\text{Coefficient of } \wp^2(x)\} \div \{\text{Coefficient of } \wp^3(x)\}$$
$$= \frac{1}{4}\{\wp'(z) - \wp'(y)\}^2\{\wp(z) - \wp(y)\}^{-2},$$

and thus we have

$$\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right\}^2 - \wp(z) - \wp(y),$$

which is a new form of the addition-theorem.

Example 1. Prove that the expression

$$\tfrac{1}{4}\{\wp'(z) - \wp'(y)\}^2\{\wp(z) - \wp(y)\}^{-2} - \wp(z) - \wp(z+y),$$

considered as a function of z, has no singularities: and deduce the addition-theorem for $\wp(z)$.

For the given expression, from the mode of its formation, can clearly have no singularities except at the points $z=0$, $z=y$, $z=-y$, and points congruent with these.

Consider then first the neighbourhood of the point $z=0$. The expression can be expanded in the form

$$\tfrac{1}{4}\{-2z^{-3} - \wp'(y) + \tfrac{1}{10}g_2 z + \ldots\}^2\{z^{-2} - \wp(y) + \tfrac{1}{20}g_2 z^2 + \ldots\}^{-2} - z^{-2} - \tfrac{1}{20}g_2 z^2 - \ldots$$
$$- \wp(y) - z\wp'(y) - \ldots,$$

and this on reduction is found to contain no negative powers of z, the first non-zero term being $\wp(y)$. The expression has therefore no singularity at the point $z=0$.

Considering next the neighbourhood of the point $z=y$, we take $z=y+x$; the expression becomes

$$\tfrac{1}{4}\{\wp'(y)+x\wp''(y)+\ldots-\wp'(y)\}^2\{\wp(y)+x\wp'(y)+\ldots-\wp(y)\}^{-2}-\wp(y)-x\wp'(y)-\ldots$$
$$-\wp(2y)-x\wp'(2y)-\ldots,$$

and this on reduction is found to contain no negative powers of x; there is therefore no singularity at the point $z=y$.

The case of the point $z=-y$ can be similarly treated.

The given expression has therefore no singularities, and so by Liouville's theorem is independent of z. But its value at the point $z=0$ has been shewn to be $\wp(y)$. We have therefore, for all values of z,

$$\tfrac{1}{4}\{\wp'(z)-\wp'(y)\}^2\{\wp(z)-\wp(y)\}^{-2}-\wp(z)-\wp(z+y)-\wp(y)=0,$$

which is the addition-theorem.

Example 2. Shew that

$$\wp(z+y)+\wp(z-y)=\{\wp(z)-\wp(y)\}^{-2}[\{2\wp(z)\wp(y)-\tfrac{1}{2}g_2\}\{\wp(z)+\wp(y)\}-g_3].$$

For by the addition-theorem we have

$$\wp(z+y)+\wp(z-y)=\frac{1}{4}\left\{\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\right\}^2-\wp(z)-\wp(y)+\frac{1}{4}\left\{\frac{\wp'(z)+\wp'(y)}{\wp(z)-\wp(y)}\right\}^2-\wp(z)-\wp(y)$$
$$=\frac{1}{2}\frac{\wp'^2(z)+\wp'^2(y)}{\{\wp(z)-\wp(y)\}^2}-2\{\wp(z)+\wp(y)\}.$$

Replacing $\wp'^2(z)$ by $4\wp^3(z)-g_2\wp(z)-g_3$, and replacing $\wp'^2(y)$ by $4\wp^3(y)-g_2\wp(y)-g_3$, and reducing, we obtain the required result.

182. *The roots* e_1, e_2, e_3.

Let Ω denote any one of the periods of $\wp(z)$, namely the quantities $2\omega_1, 2\omega_2, 2\omega_1+2\omega_2, 2\omega_1-2\omega_2, -2\omega_1-2\omega_2, \ldots$. Then

$$\wp'\left(\frac{1}{2}\Omega\right)=\wp'\left(\frac{1}{2}\Omega-\Omega\right), \text{ since } \wp'(z) \text{ has the period } \Omega,$$
$$=\wp'\left(-\frac{1}{2}\Omega\right)$$
$$=-\wp'\left(\frac{1}{2}\Omega\right), \text{ since } \wp' \text{ is an odd function of } z.$$

It follows from this that unless $\frac{1}{2}\Omega$ is itself a period (in which case $\wp'\left(\frac{1}{2}\Omega\right)$ is infinite), $\wp'\left(\frac{1}{2}\Omega\right)$ is zero.

We have therefore

$$\wp'(\omega_1)=0,\quad \wp'(\omega_2)=0,\quad \wp'(\omega_3)=0,$$

where ω_3 stands for $-(\omega_1+\omega_2)$.

Now denote the quantities $\wp(\omega_1)$, $\wp(\omega_2)$, $\wp(\omega_3)$ by e_1, e_2, e_3, respectively. Then the equation

$$\wp'^2(\omega_1) = 4\wp^3(\omega_1) - g_2\wp(\omega_1) - g_3,$$

or

$$0 = 4e_1^3 - g_2 e_1 - g_3,$$

shews that e_1 is a root of the cubic equation

$$4\wp^3 - g_2\wp - g_3 = 0.$$

Similarly e_2 and e_3 are roots of this equation.

Moreover, the quantities e_1, e_2, e_3 are *distinct* roots of the equation; for if for example we had $e_1 = e_3$, we should have $\wp(\omega_3) = \wp(\omega_1)$, and therefore

$$\omega_3 = \pm\,\omega_1 + \text{a period},$$

which is not the case.

We see therefore that *the three roots of the cubic*

$$4\wp^3 - g_2\wp - g_3 = 0$$

are e_1, e_2, e_3, *where*

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3),$$

and

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

The quantities e_1, e_2, e_3 therefore satisfy the relations

$$e_1 + e_2 + e_3 = 0,$$

$$e_2e_3 + e_3e_1 + e_1e_2 = -\frac{1}{4}g_2,$$

$$e_1e_2e_3 = \frac{1}{4}g_3.$$

183. *Addition of a half-period to the argument of* $\wp(z)$.

From the addition-theorem we have

$$\wp(z+\omega_1) + \wp(z) + e_1 = \frac{1}{4}\wp'^2(z)\{\wp(z) - e_1\}^{-2}$$

$$= \{\wp(z) - e_1\}\{\wp(z) - e_2\}\{\wp(z) - e_3\}\{\wp(z) - e_1\}^{-2}$$

$$= \{\wp(z) - e_2\}\{\wp(z) - e_3\}\{\wp(z) - e_1\}^{-1},$$

or

$$\wp(z+\omega_1) = e_1 + (e_1 - e_2)(e_1 - e_3)\{\wp(z) - e_1\}^{-1}.$$

This formula expresses the result of adding a half-period to the argument of the Weierstrassian elliptic function.

Example 1. Shew that

$$\wp'(z)\,\wp'(z+\omega_1)\,\wp'(z+\omega_2)\,\wp'(z+\omega_3)$$

is a multiple of the discriminant of the equation

$$4x^3-g_2x-g_3=0.$$

For we have

$$\wp(z+\omega_1)-e_1=(e_1-e_2)(e_1-e_3)\{\wp(z)-e_1\}^{-1}.$$

Differentiating, we have

$$\wp'(z+\omega_1)=-(e_1-e_2)(e_1-e_3)\,\wp'(z)\{\wp(z)-e_1\}^{-2}.$$

Therefore

$$\begin{aligned}&\wp'(z)\,\wp'(z+\omega_1)\,\wp'(z+\omega_2)\,\wp'(z+\omega_3)\\&\quad=(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2\,\wp'^4(z)\{\wp(z)-e_1\}^{-2}\{\wp(z)-e_2\}^{-2}\{\wp(z)-e_3\}^{-2}\\&\quad=16(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2,\end{aligned}$$

which is a multiple of the discriminant of the equation

$$4(x-e_1)(x-e_2)(x-e_3)=0.$$

Example 2. Shew that

$$\{\wp(2z)-e_1\}\{\wp(2z)-e_2\}+\{\wp(2z)-e_1\}\{\wp(2z)-e_3\}+\{\wp(2z)-e_2\}\{\wp(2z)-e_3\}=\wp(z)-\wp(2z).$$

184. *Integration of* $(ax^4+4bx^3+6cx^2+4dx+e)^{-\frac{1}{2}}$.

We shall now shew how certain problems in the Integral Calculus, whose solution cannot be found in terms of the elementary functions, can be solved by aid of the function $\wp(z)$.

Let the general quartic polynomial be written

$$f(x)=ax^4+4bx^3+6cx^2+4dx+e.$$

Let its invariants* be

$$g_2=ae-4bd+3c^2,$$

$$g_3=\begin{vmatrix}a & b & c\\ b & c & d\\ c & d & e\end{vmatrix}=ace+2bcd-c^3-ad^2-b^2e;$$

let its Hessian be

$$\begin{aligned}h(x)=(ac-b^2)x^4+2(ad-bc)x^3+(ae+2bd-3c^2)x^2\\+2(be-cd)x+(ce-d^2),\end{aligned}$$

and let its sextic covariant be

$$\begin{aligned}t(x)&=\tfrac{1}{2}\{-f(x)h'(x)+h(x)f'(x)\}\\&=(a^2d-3abc+2b^3)x^6+\ldots.\end{aligned}$$

* The student who is not already familiar with the elements of the theory of binary forms is referred to Burnside and Panton's *Theory of Equations*, where the invariants and covariants of the quartic are discussed.

Then it is known that

$$t^2(x) = -4h^3(x) + g_2 f^2(x) h(x) - g_3 f^3(x).$$

If we write $s = -h(x)/f(x)$, this relation becomes

$$t^2(x) = f^3(x)(4s^3 - g_2 s - g_3).$$

Now
$$ds = \frac{h(x) f'(x) - h'(x) f(x)}{f^2(x)} dx$$
$$= \frac{2t(x)}{f^2(x)} dx,$$

and so
$$(4s^3 - g_2 s - g_3)^{-\frac{1}{2}} ds = 2\{f(x)\}^{-\frac{1}{2}} dx.$$

Let x_0 be any root of the equation $f(x) = 0$; then to the value $x = x_0$ corresponds $s = \infty$; and hence, if we write

$$z = \int_{x_0}^{x} \{f(x)\}^{-\frac{1}{2}} dx,$$

we have
$$2z = \int_{\infty}^{s} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt.$$

It follows that *the equation*

$$\wp(2z;\ g_2,\ g_3) = -h(x)/f(x)$$

is an integrated form of the equation

$$z = \int_{x_0}^{x} \{ax^4 + 4bx^3 + 6cx^2 + 4dx + e\}^{-\frac{1}{2}} dx.$$

Example 1. Shew that (with the same notation)

$$\wp'(2z;\ g_2, g_3) = \mp t(x)\{f(x)\}^{-\frac{3}{2}}.$$

Example 2. Shew also that, if

$$y = \int_{x_0}^{u} \{f(t)\}^{-\frac{1}{2}} dt,$$

then $\wp(z+y)$ and $\wp(z-y)$ are the roots of the equation

$$(x-u)^2 \wp^2 - F(x, u)\wp - H(x, u) - \tfrac{1}{12} g_2 (x-u)^2 = 0,$$

where
$$F(x, u) = ax^2u^2 + 2bxu(x+u) + c(x^2 + 4xu + u^2) + 2d(x+u) + e,$$

and $H(x, u)$ is derived from $h(x)$ in the same way as $F(x, u)$ from $f(x)$.

(Cambridge Mathematical Tripos, Part II, 1896.)

185. *Another solution of the integration-problem.*

The integration discussed in the last article may also be effected in the following way.

As before, let

$$z = \int_{x_0}^{x} \{f(x)\}^{-\frac{1}{2}} dx,$$

where $$f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$
and let x_0 be a root of the equation $f(x) = 0$.

Then, by Taylor's theorem, we have

$$f(x) = (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \frac{1}{6}(x - x_0)^3 f'''(x_0) + \frac{1}{24}(x - x_0)^4 f''''(x_0)$$

Writing $(x - x_0)^{-1} = \zeta$, we have

$$f(x) = \zeta^{-4}\left\{f'(x_0)\zeta^3 + \frac{1}{2}f''(x_0)\zeta^2 + \frac{1}{6}f'''(x_0)\zeta + \frac{1}{24}f''''(x_0)\right\},$$

and so $$z = \int_\zeta^\infty \left\{f'(x_0)\zeta^3 + \frac{1}{2}f''(x_0)\zeta^2 + \frac{1}{6}f'''(x_0)\zeta + \frac{1}{24}f''''(x_0)\right\}^{-\frac{1}{2}} d\zeta.$$

Writing $\zeta = 4\{f'(x_0)\}^{-1}\theta$, we have

$$z = \int_\theta^\infty \left\{4\theta^3 + \frac{1}{2}f''(x_0)\theta^2 + \frac{1}{24}f'(x_0)f'''(x_0)\theta + \frac{1}{24.16}f'^2(x_0)f''''(x_0)\right\}^{-\frac{1}{2}} d\theta.$$

Now take a new variable of integration s, defined by the equation

$$\theta = s - \frac{1}{24}f''(x_0);$$

this substitution destroys the term involving the square of the variable of integration in the denominator, and we thus have

$$z = \int_s^\infty \{4s^3 - g_2 s - g_3\}^{-\frac{1}{2}} ds,$$

where

$$g_2 = \frac{1}{48}f''^2(x_0) - \frac{1}{24}f'(x_0)f'''(x_0),$$

$$g_3 = \frac{1}{24^2}\left\{f'(x_0)f''(x_0)f'''(x_0) - \frac{1}{3}f''^3(x_0) - \frac{3}{2}f'^2(x_0)f''''(x_0)\right\}.$$

It can easily be verified that these latter quantities are the same as the invariants g_2 and g_3 of the last article.

We have therefore

$$s = \wp(z;\ g_2, g_3),$$

and therefore $$\theta = \wp(z) - \frac{1}{24}f''(x_0),$$

$$\zeta = 4\{f'(x_0)\}^{-1}\left\{\wp(z) - \frac{1}{24}f''(x_0)\right\},$$

and finally $$x = x_0 + \frac{1}{4}f'(x_0)\left\{\wp(z) - \frac{1}{24}f''(x_0)\right\}^{-1}.$$

This last equation is the integral-equivalent of the equation

$$z = \int_{x_0}^x \{f(x)\}^{-\frac{1}{2}} dx.$$

It may be observed that

$$\wp'(z) = (4s^3 - g_2 s - g_3)^{\frac{1}{2}} = \frac{1}{4} f'(x_0) \{f(x)\}^{\frac{1}{2}} \zeta^2,$$

and hence that

$$\{f(x)\}^{\frac{1}{2}} = \frac{f'(x_0)\wp'(z)}{4\left\{\wp(z) - \frac{1}{24} f''(x_0)\right\}^2}.$$

Example. Shew that the integrated form of the equation

$$z = \int_{x_0}^{x} \{f(x)\}^{-\frac{1}{2}} dx,$$

where x_0 is any constant (not necessarily a root of $f(x)$), and $f(x)$ is any quartic function of x, is

$$x = x_0 + \frac{f^{\frac{1}{2}}(x_0)\wp'(z) + \frac{1}{2}f'(x_0)\{\wp(z) - \frac{1}{24}f''(x_0)\} + \frac{1}{24}f(x_0)f'''(x_0)}{2\{\wp(z) - \frac{1}{24}f''(x_0)\}^2 - \frac{1}{48}f(x_0)f^{iv}(x_0)},$$

where $\wp$ is the Weierstrassian elliptic function formed with the invariants g_2 and g_3 of $f(x)$.

Shew further that

$$\wp(z) = \frac{f^{\frac{1}{2}}(x)f^{\frac{1}{2}}(x_0) + f(x_0)}{2(x-x_0)^2} + \frac{f'(x_0)}{4(x-x_0)} + \frac{1}{24}f''(x_0)$$

and

$$\wp'(z) = \left\{\frac{f(x)}{(x-x_0)^3} - \frac{1}{4}\frac{f'(x)}{(x-x_0)^2}\right\} f^{\frac{1}{2}}(x_0) - \left\{\frac{f(x_0)}{(x-x_0)^3} - \frac{1}{4}\frac{f'(x_0)}{(x-x_0)^2}\right\} f^{\frac{1}{2}}(x).$$

186. *Uniformisation of curves of genus unity.*

The theorem of the last article may be stated somewhat differently thus:

If two variables y and x are connected by an equation of the form

$$y^2 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$

then it is possible to express them in terms of a third variable z by means of the equations

$$\begin{cases} x = x_0 + \frac{1}{4} f'(x_0) \left\{\wp(z) - \frac{1}{24} f''(x_0)\right\}^{-1}, \\ y = \frac{1}{4} f'(x_0) \wp'(z) \left\{\wp(z) - \frac{1}{24} f''(x_0)\right\}^{-2}, \end{cases}$$

where $$f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$

x_0 is any root of the equation $f(x) = 0$, and the function $\wp(z)$ is formed with the invariants g_2 and g_3 of the quartic $f(x)$; moreover, the quantity z is defined by the equation

$$z = \int_{x_0}^{x} \{f(x)\}^{-\frac{1}{2}} dx.$$

Now y is a two-valued function of x, since the quantity

$$\pm (ax^4 + 4bx^3 + 6cx^2 + 4dx + e)^{\frac{1}{2}}$$

may take either sign; and x is a four-valued function of y, since the equation in x

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + (e - y^2) = 0$$

has four roots. But on referring to the equations which express x and y in terms of z, we see that *x and y are one-valued functions of z.* It is this fact which gives importance to the variable z; z is called the *uniformising variable* of the equation

$$y^2 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e.$$

The student who is acquainted with the theory of algebraic plane curves will be aware that curves are classified according to their *genus**, a number which may be geometrically interpreted as the difference between the number of double points possessed by the curve and the maximum number of double points which can be possessed by a curve of the same degree as the given curve. Curves whose genus is zero are called *unicursal curves*; if $f(x, y)=0$ is the equation of a unicursal curve, it is known that x and y can be expressed in the form

$$\begin{cases} x = \phi(z) \\ y = \psi(z) \end{cases},$$

where ϕ and ψ are rational functions of their argument; since rational functions are always one-valued, it follows that the variable z thus introduced is the *uniformising variable* for the equation $f(x, y)=0$; i.e., although y is in general a many-valued function of x, and x is a many-valued function of y, yet x and y are one-valued functions of z.

Considering now curves whose genus is not zero, let

$$f(x, y) = 0$$

be a curve of genus unity. Then it can be shewn that x and y can be expressed in the form

$$\begin{cases} x = \phi(z) \\ y = \psi(z) \end{cases},$$

where ϕ and ψ are now elliptic functions of their argument z; x and y are thus expressed as one-valued functions of z, and z is the uniformising variable of the equation $f(x, y)=0$. This result is obtained by writing

$$\begin{cases} x = F(\xi, \eta) \\ y = G(\xi, \eta) \end{cases},$$

where F and G are rational functions of their arguments, and choosing F and G in such a way that the equation $f(x, y)=0$ is transformed into an equation of the form

$$\eta^2 - (4\xi^3 - g_2\xi - g_3) = 0;$$

we can then write

$$\begin{cases} \xi = \wp(z;\, g_2, g_3) \\ \eta = \wp'(z;\, g_2, g_3) \end{cases},$$

and x and y will thus be expressed as one-valued functions of z.

When the genus of the algebraic curve

$$f(x, y) = 0$$

is greater than unity, the uniformisation can be effected by means of *automorphic functions.* Two classes of automorphic functions are known by which this uniformisation

* In French *genre*, in German *Geschlecht*.

may be effected: namely, one which was first given by Weber in *Göttinger Nachrichten*, 1886, and one which was first given by the author, *Phil. Trans.*, 1898. In the case of Weber's functions, the "fundamental polygon" (the analogue of the period-parallelogram) is "multiply-connected," i.e. consists of a region containing islands which are to be regarded as not belonging to it. In the case of the functions described in *Phil. Trans.*, the fundamental-polygon is "simply-connected," i.e. is the area enclosed by a polygon. This latter class of functions may be regarded as the immediate generalisation of elliptic functions.

Miscellaneous Examples.

1. Shew that

$$\wp(z+y)-\wp(z-y)=-\wp'(z)\,\wp'(y)\,\{\wp(z)-\wp(y)\}^{-2}.$$

2. Prove that

$$\wp(z)-\wp(z+y+w)=2\frac{\partial}{\partial z}\frac{\Sigma\wp^2(z)\,\{\wp(y)-\wp(w)\}}{\Sigma\wp'(z)\,\{\wp(y)-\wp(w)\}},$$

where, on the right-hand side, the subject of differentiation is symmetrical in z, y, and w.

(Cambridge Mathematical Tripos, Part I, 1897.)

3. Shew that

$$\begin{vmatrix} \wp'''(z-y) & \wp'''(y-w) & \wp'''(w-z) \\ \wp''(z-y) & \wp''(y-w) & \wp''(w-z) \\ \wp(z-y) & \wp(y-w) & \wp(w-z) \end{vmatrix} = \tfrac{1}{2}g_2 \begin{vmatrix} \wp'''(z-y) & \wp'''(y-w) & \wp'''(w-z) \\ \wp(z-y) & \wp(y-w) & \wp(w-z) \\ 1 & 1 & 1 \end{vmatrix}.$$

(Trinity College Scholarship Examination, 1898.)

4. If

$$y=\wp(z)-e_1,\quad y'=\frac{dy}{dz},$$

simplify the expression

$$\left\{y'\left(y-\frac{1}{4}\frac{d^2}{dz^2}\log y'\right)^{\frac{1}{2}}+(e_1-e_2)(e_1-e_3)\right\}^{\frac{1}{2}},$$

where e_1, e_2, e_3 are the values of $\wp(z)$ for which $\wp'(z)=0$.

(Cambridge Mathematical Tripos, Part I, 1897.)

5. Prove that

$$\Sigma\,\{\wp(z)-e\}\,\{\wp(y)-\wp(w)\}^2\,\{\wp(y+w)-e\}^{\frac{1}{2}}\,\{\wp(y-w)-e\}^{\frac{1}{2}}=0,$$

where the sign of summation refers to any three arguments z, y, w, and e is any one of the quantities e_1, e_2, e_3.

(Cambridge Mathematical Tripos, Part I, 1896.)

6. Shew that

$$\frac{\wp'(z+\omega_1)}{\wp'(z)}=-\left\{\frac{\wp(\frac{1}{2}\omega_1)-\wp(\omega_1)}{\wp(z)-\wp(\omega_1)}\right\}^2.$$

(Cambridge Mathematical Tripos, Part I, 1894.)

7. Prove that

$$\wp(2z)-\wp(\omega_1)=\{\wp'(z)\}^{-2}\,\{\wp(z)-\wp(\tfrac{1}{2}\omega_1)\}^2\,\{\wp(z)-\wp(\omega_2+\tfrac{1}{2}\omega_1)\}^2.$$

(Cambridge Mathematical Tripos, Part I, 1894.)

8. If m be any constant, prove that

$$\frac{1}{\wp'(y)}\int\frac{e^{m\{\wp(z)-\wp(y)\}}\wp'^2(z)\,dz}{\wp(z)-\wp(y)}+\wp'(z)\int\frac{e^{m\{\wp(z)-\wp(y)\}}\,dy}{\wp(z)-\wp(y)}$$

$$=-\frac{1}{2}\Sigma\iint\frac{e^{m\{\wp(z)-\wp(y)\}}\wp'^2(z)\,dz\,dy}{\{\wp(z)-e_1\}\{\wp(y)-e_1\}},$$

where the summation refers to the values of $\wp(z)$ for which $\wp'(z)$ is zero; and the integrals are indefinite.

(Cambridge Mathematical Tripos, Part I, 1897.)

9. Let

$$R(x)=Ax^4+Bx^3+Cx^2+Dx+E,$$

and let $\xi=\phi(x)$ be the function defined by the equation

$$x=\int^{\xi}\{R(\xi)\}^{-\frac{1}{2}}\,d\xi,$$

where the lower limit of the integral is arbitrary. Shew that

$$\frac{2\phi'(a)}{\phi(x+y)-\phi(a)}=\frac{\phi'(a+y)+\phi'(a)}{\phi(a+y)-\phi(a)}+\frac{\phi'(a-y)+\phi'(a)}{\phi(a-y)-\phi(a)}-\frac{\phi'(a+y)-\phi'(x)}{\phi(a+y)-\phi(x)}$$

$$-\frac{\phi'(a-y)-\phi'(x)}{\phi(a-y)-\phi(x)}.$$

(Hermite.)

10. Shew that when the change of variables

$$\xi'=\frac{\xi}{\eta},\quad \eta'=\frac{\xi^3}{\eta^2}$$

is applied to the equations

$$\begin{cases}\eta^2+\eta(1+p\xi)+\xi^3=0,\\[2ex] du-\dfrac{d\xi}{2\eta+1+p\xi}=0,\end{cases}$$

they transform into the similar equations

$$\begin{cases}\eta'^2+\eta'(1+p\xi')+\xi'^3=0,\\[2ex] du-\dfrac{d\xi'}{2\eta'+1+p\xi'}=0.\end{cases}$$

Shew that the result of performing this change of variables three times in succession is a return to the original variables ξ, η; and hence prove that if ξ and η be denoted as functions of u by $E(u)$ and $F(u)$ respectively, then

$$E(u+A)=\frac{E(u)}{F(u)},$$

$$F(u+A)=\frac{E^3(u)}{F^2(u)},$$

where A is one-third of a period of the functions $E(u)$ and $F(u)$.

Shew that
$$E(u)=\frac{p^2}{12}-\wp(u\,;\,g_2,\,g_3),$$

where
$$g_2=2p+\frac{1}{12}p^4,\quad g_3=-1-\frac{1}{6}p^3-\frac{1}{216}p^6.$$

(De Brun.)

CHAPTER XV.

THE ELLIPTIC FUNCTIONS sn z, cn z, dn z.

187. *Construction of a doubly-periodic function with two simple poles in each period-parallelogram.*

The function $\wp(z)$, which has been considered in the previous chapter, is a doubly-periodic function of z, with a single pole of the second order in each period-parallelogram, namely at the point congruent with the origin*. We shall next introduce a doubly-periodic function which differs from $\wp(z)$ in having *two* poles, each *simple*, in every period-parallelogram.

Consider the series

$$f(z) = \Sigma\left[\{z + 2m\omega_1 + (2n+1)\omega_2\}^{-1} - \{2m\omega_1 + (2n+1)\omega_2\}^{-1}\right.$$
$$\left. - \{z + (2m+1)\omega_1 + (2n+1)\omega_2\}^{-1} + \{(2m+1)\omega_1 + (2n+1)\omega_2\}^{-1}\right],$$

in which the summation extends over all positive and negative integer and zero values of m and n.

When the modulus of $(2m\omega_1 + 2n\omega_2)$ is large (and we may suppose the series arranged in order of ascending values of $|2m\omega_1 + 2n\omega_2|$), the terms of the series bear a ratio of approximate equality to those of the series

$$\Sigma\left[-z\{2m\omega_1 + (2n+1)\omega_2\}^{-2} + z\{(2m+1)\omega_1 + (2n+1)\omega_2\}^{-2}\right],$$

or
$$-z\Sigma\{2m\omega_1 + (2n+1)\omega_2\}^{-2}\left[1 - \left\{1 + \frac{\omega_1}{2m\omega_1 + (2n+1)\omega_2}\right\}^{-2}\right],$$

and these terms bear a ratio of approximate equality to those of the series

$$-2z\omega_1\Sigma\{2m\omega_1 + (2n+1)\omega_2\}^{-3},$$

which again bear a finite ratio to those of the series

$$\Sigma(2m\omega_1 + 2n\omega_2)^{-3},$$

which was shewn in § 11 to be an absolutely convergent series.

* In the network of parallelograms described in § 177, the poles of $\wp(z)$ are not within the parallelograms, but on their bounding lines. We may however suppose the whole network slightly translated so as to bring the poles within the parallelograms.

It follows that the series which represents $f(z)$ is absolutely convergent for all values of z, except for the exceptional values included in the formula

$$z = m\omega_1 + (2n+1)\,\omega_2, \qquad (m, n, \text{ integers})$$

for which the several terms of the series are infinite, and which have been tacitly excluded from the foregoing discussion of convergence.

Moreover, since the terms of the comparison-series are independent of z, the convergence is (§ 52) not only absolute but uniform.

By a discussion similar to that in § 176, we can shew that $f(z)$ is a *doubly-periodic* function of z, whose periods are $2\omega_1$ and $2\omega_2$; it is an *odd* function of z, so that

$$f(z) = -f(-z);$$

and its *singularities* are at the points

$$z = m\omega_1 + (2n+1)\,\omega_2,$$

where m and n may have any integer or zero values; these singularities are simple poles, with the residues ± 1. There are two of these singularities in each period-parallelogram.

188. *Expression of the function $f(z)$ by means of an integral.*

The singularities of $f(z)$ in the fundamental period-parallelogram are, as we have seen, at the points $z = \omega_2$ and $z = \omega_1 + \omega_2$.

Consider now the neighbourhood of the point $z = \omega_2$.

Writing $z = \omega_2 + x$, we have

$$\begin{aligned} f(\omega_2 + x) &= -f(-\omega_2 - x), \quad \text{since } f \text{ is an odd function,} \\ &= -f(2\omega_2 - \omega_2 - x), \quad \text{since } 2\omega_2 \text{ is a period,} \\ &= -f(\omega_2 - x), \end{aligned}$$

from which it follows that $f(\omega_2 + x)$ is an odd function of x; the expansion of $f(z)$ in ascending powers of x will therefore contain only odd powers of x.

Now

$$\begin{aligned} f(z) = \Sigma\,[\{x + 2m\omega_1 + (2n+2)\,\omega_2\}^{-1} - \{2m\omega_1 + (2n+1)\,\omega_2\}^{-1} \\ - \{x + (2m+1)\,\omega_1 + (2n+2)\,\omega_2\}^{-1} + \{(2m+1)\,\omega_1 + (2n+1)\,\omega_2\}^{-1}], \end{aligned}$$

where the summation extends over all positive and negative integer and zero values of m and n.

In this expression, replace all expressions of the form $(A+x)^{-1}$ by their expansions $A^{-1} - A^{-2}x + A^{-3}x^2 - \ldots$, x being supposed small. A term

in x^{-1} will arise from the pair of values $(m=0,\ n=-1)$, and we thus have

$$f(z) = \frac{1}{x} + Bx + Cx^3 + \dots,$$

where $$B = \Sigma\left[-\{2m\omega_1 + 2n\omega_2\}^{-2} + \{(2m+1)\,\omega_1 + 2n\omega_2\}^{-2}\right],$$

the summation being in this case extended over all positive and negative integer and zero values of m and n, excluding simultaneous zeros in the first term.

If now by means of this expansion we express the quantity

$$f'^2(z) - f^4(z) + 6Bf^2(z)$$

as a series of powers of x, it is found that the negative powers of x destroy each other; this quantity has therefore no singularity at the point $z = \omega_2$.

Consider next the neighbourhood of the point $z = \omega_1 + \omega_2$.

Writing $z = \omega_1 + \omega_2 + y$, we have

$$f(\omega_1 + \omega_2 + y) = -f(-\omega_1 - \omega_2 - y), \quad \text{since } f \text{ is an odd function,}$$

$$= -f(\omega_1 + \omega_2 - y), \quad \text{since } (2\omega_1 + 2\omega_2) \text{ is a period.}$$

It follows that $f(\omega_1 + \omega_2 + y)$ is an odd function of y; its expansion in powers of y will therefore contain only odd powers of y.

Now expanding $f(z)$ in powers of y, in the same way as $f(z)$ was formerly expanded in powers of x, we find that

$$f(z) = -\frac{1}{y} + B'y + C'y^3 + \dots,$$

where $$B' = \Sigma\left[-\{(2m-1)\,\omega_1 + 2n\omega_2\}^{-2} + \{2m\omega_1 + 2n\omega_2\}^{-2}\right],$$

the summation extending over all positive and negative integer and zero values of m and n, excluding simultaneous zeros in the second term.

Comparing this with the expansion of B, we have

$$B' = -B,$$

so $$f(z) = -\frac{1}{y} - By + C'y^3 + \dots,$$

and, as before, the quantity

$$f'^2(z) - f^4(z) + 6Bf^2(z)$$

has no singularity at the point $z = \omega_1 + \omega_2$.

Now the points $z = \omega_2$ and $z = \omega_1 + \omega_2$ are the only possible singularities of this quantity in the period-parallelogram; it has therefore no singularity in the parallelogram, and therefore (since it is doubly-periodic) no singularities

in the whole z-plane; it is therefore by Liouville's theorem (§ 47) a constant independent of z, say A.

The function $f(z)$ therefore satisfies a differential equation

$$f'^2(z) = f^4(z) - 6Bf^2(z) + A.$$

Replacing B and A by new constants k and μ, we can write this in the form

$$f'^2(z) = \left\{\frac{k^2}{\mu^2} - f^2(z)\right\}\left\{\frac{1}{\mu^2} - f^2(z)\right\};$$

so that, as $f(z)$ is zero when z is zero,

$$z = \int_0^{f(z)} \left\{\frac{k^2}{\mu^2} - t^2\right\}^{-\frac{1}{2}} \left\{\frac{1}{\mu^2} - t^2\right\}^{-\frac{1}{2}} dt.$$

We see therefore that *the odd doubly-periodic function $f(z)$, which has periods $2\omega_1$ and $2\omega_2$ and simple poles at all points congruent with $z = \omega_1$ and $z = \omega_1 + \omega_2$, may be regarded as defined by the equation*

$$z = \int_0^{f(z)} \left\{\frac{k^2}{\mu^2} - t^2\right\}^{-\frac{1}{2}} \left\{\frac{1}{\mu^2} - t^2\right\}^{-\frac{1}{2}} dt,$$

where k and μ are constants depending only on ω_1 and ω_2.

189. *The function* sn z.

The function $f(z)$ discussed in the last two articles can be expressed in terms of another function, which we shall denote by sn z, in the following way.

Replacing the variable t of integration by a new variable s, defined by the equation $ks = \mu t$, we have

$$z = \mu \int_0^{\frac{\mu}{k} f(z)} (1 - s^2)^{-\frac{1}{2}} (1 - k^2 s^2)^{-\frac{1}{2}} ds.$$

Now define the new function sn z by the relation

$$\mu f(\mu z) = k \operatorname{sn} z;$$

then we have

$$z = \int_0^{\operatorname{sn} z} (1 - s^2)^{-\frac{1}{2}} (1 - k^2 s^2)^{-\frac{1}{2}} ds.$$

This last equation can be regarded as the definition of the function sn z in terms of its argument z and the constant-parameter k, which is called the *modulus*; it is analogous to the definition of the function $\sin z$ by the relation

$$z = \int_0^{\sin z} (1 - s^2)^{-\frac{1}{2}} ds.$$

From the equation

$$\mu f(\mu z) = k \operatorname{sn} z,$$

it is clear that the function $\operatorname{sn} z$ has the same general properties as $f(z)$, namely, it is an odd one-valued doubly-periodic function of z, with two poles in each period-parallelogram, the distance between the poles being half of one of the periods. The two periods will be connected by a relation, as they depend only on the single constant k.

190. *The functions* $\operatorname{cn} z$ *and* $\operatorname{dn} z$.

We now proceed to introduce two other functions, either of which may be regarded as bearing to the function $\operatorname{sn} z$ a relation similar to that which the function $\cos z$ bears to $\sin z$.

Since
$$z = \int_0^{\operatorname{sn} z} (1 - s^2)^{-\frac{1}{2}} (1 - k^2 s^2)^{-\frac{1}{2}} \, ds,$$

we have
$$\frac{dz}{d(\operatorname{sn} z)} = (1 - \operatorname{sn}^2 z)^{-\frac{1}{2}} (1 - k^2 \operatorname{sn}^2 z)^{-\frac{1}{2}},$$

or
$$\frac{d}{dz}(\operatorname{sn} z) = (1 - \operatorname{sn}^2 z)^{\frac{1}{2}} (1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}.$$

Now $\operatorname{sn} z$ is a one-valued function of z, so its derivate must be also a one-valued function. It follows that

$$(1 - \operatorname{sn}^2 z)^{\frac{1}{2}} (1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$$

can have no branch-points (§ 46), considered as a function of z; and therefore either

(α) Each of the quantities $(1 - \operatorname{sn}^2 z)^{\frac{1}{2}}$ and $(1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$ is a function of z which has no branch-points, or

(β) The functions $(1 - \operatorname{sn}^2 z)^{\frac{1}{2}}$ and $(1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$ have branch-points, but are such that their product has no branch-points.

Now the alternative (β) could be true only if the functions $(1 - \operatorname{sn}^2 z)^{\frac{1}{2}}$ and $(1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$ had their branch-points at the same places; but this is not the case, since $(1 - \operatorname{sn}^2 z)^{\frac{1}{2}}$ has branch-points at the places when $\operatorname{sn}^2 z = 1$, and $(1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$ has not. The alternative (β) being thus ruled out, we see that the alternative (α) must hold.

If now we write

$$\operatorname{cn} z = (1 - \operatorname{sn}^2 z)^{\frac{1}{2}},$$

$$\operatorname{dn} z = (1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}},$$

where it is supposed that each of these functions has the value unity when $\operatorname{sn} z$ is zero, then since $\operatorname{cn} z$ and $\operatorname{dn} z$ have no branch-points, and have definite values at the point $z = 0$, it follows that *the functions* $\operatorname{cn} z$ *and* $\operatorname{dn} z$ *are one-valued functions of* z.

They obviously satisfy the relations

$$\operatorname{sn}^2 z + \operatorname{cn}^2 z = 1,$$
$$k^2 \operatorname{sn}^2 z + \operatorname{dn}^2 z = 1.$$

The functions $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$ are often called the *Jacobian elliptic functions.*

The function $\cos z$ is in the same way a one-valued function, although the occurrence of the radical in $(1-\sin^2 z)^{\frac{1}{2}}$ might lead us at first sight to suppose that it possessed branch-points.

191. *Expression of* cn z *and* dn z *by means of integrals.*

We shall next find, for the functions $\operatorname{cn} z$ and $\operatorname{dn} z$, integral-expressions similar to that found in § 189 for $\operatorname{sn} z$.

Differentiating the equation

$$\operatorname{cn}^2 z = 1 - \operatorname{sn}^2 z,$$

we have
$$\operatorname{cn} z \frac{d}{dz} \operatorname{cn} z = - \operatorname{sn} z \operatorname{cn} z \operatorname{dn} z,$$

so
$$\frac{d}{dz} \operatorname{cn} z = - \operatorname{sn} z \operatorname{dn} z$$
$$= - \{(1 - \operatorname{cn}^2 z)(k'^2 + k^2 \operatorname{cn}^2 z)\}^{\frac{1}{2}},$$

where
$$k'^2 = 1 - k^2.$$

Thus if $\operatorname{cn} z = t$, we have

$$dz = -(1-t^2)^{-\frac{1}{2}} (k'^2 + k^2 t^2)^{-\frac{1}{2}}\, dt,$$

and therefore (since $\operatorname{cn} z = 1$ when $z = 0$)

$$z = \int_{\operatorname{cn} z}^{1} (1-t^2)^{-\frac{1}{2}} (k'^2 + k^2 t^2)^{-\frac{1}{2}}\, dt.$$

In the same way we can shew that

$$\frac{d}{dz} \operatorname{dn} z = - k^2 \operatorname{sn} z \operatorname{cn} z,$$

and
$$z = \int_{\operatorname{dn} z}^{1} (1-t^2)^{-\frac{1}{2}} (t^2 - k'^2)^{-\frac{1}{2}}\, dt.$$

Example 1. If $\operatorname{cs} z = \operatorname{cn} z / \operatorname{sn} z$, shew that

$$z = \int_{\operatorname{cs} z}^{\infty} (t^2+1)^{-\frac{1}{2}} (t^2 + k'^2)^{-\frac{1}{2}}\, dt.$$

Example 2. If $\operatorname{sd} z = \operatorname{sn} z / \operatorname{dn} z$, shew that

$$z = \int_0^{\operatorname{sd} z} (1 - k'^2 t^2)^{-\frac{1}{2}} (1 + k^2 t^2)^{-\frac{1}{2}}\, dt$$

192. *The addition-theorem for the function* dn z.

We shall next shew how to find dn x, where

$$x = y + z,$$

in terms of the sn, cn, and dn, of y and z: the result will be the *addition-theorem* for the function dn.

Suppose that y and z vary, x remaining constant, so that

$$\frac{dz}{dy} = -1.$$

Introducing new variables u and v, defined by the equations

$$u = \operatorname{cn} z \operatorname{cn} y,$$

$$v = \operatorname{sn} z \operatorname{sn} y,$$

we have
$$\frac{dv}{du} = \frac{\dfrac{dv}{dy}}{\dfrac{du}{dy}} = \frac{\operatorname{sn} z \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} z \operatorname{dn} z \dfrac{dz}{dy}}{-\operatorname{cn} z \operatorname{sn} y \operatorname{dn} y - \operatorname{cn} y \operatorname{sn} z \operatorname{dn} z \dfrac{dz}{dy}},$$

or
$$\frac{dv}{du} = \frac{\operatorname{sn} z \operatorname{cn} y \operatorname{dn} y - \operatorname{sn} y \operatorname{cn} z \operatorname{dn} z}{\operatorname{cn} y \operatorname{sn} z \operatorname{dn} z - \operatorname{cn} z \operatorname{sn} y \operatorname{dn} y}.$$

From this we obtain the equations

$$\left(\frac{dv}{du}\right)^2 - 1 = k^2 (\operatorname{sn}^2 y - \operatorname{sn}^2 z)^2 (\operatorname{cn} y \operatorname{sn} z \operatorname{dn} z - \operatorname{cn} z \operatorname{sn} y \operatorname{dn} y)^{-2},$$

$$v - u\frac{dv}{du} = (\operatorname{sn} y \operatorname{cn} y \operatorname{dn} z - \operatorname{sn} z \operatorname{cn} z \operatorname{dn} y)(\operatorname{cn} y \operatorname{sn} z \operatorname{dn} z - \operatorname{cn} z \operatorname{sn} y \operatorname{dn} y)^{-1},$$

$$\left(\frac{dv}{du}\right)^2 - \left(v - u\frac{dv}{du}\right)^2 = (\operatorname{sn}^2 y - \operatorname{sn}^2 z)^2 (\operatorname{cn} y \operatorname{sn} z \operatorname{dn} z - \operatorname{cn} z \operatorname{sn} y \operatorname{dn} y)^{-2},$$

and consequently

$$\frac{1}{k^2}\left(\frac{dv}{du}\right)^2 - \frac{1}{k^2} = \left(\frac{dv}{du}\right)^2 - \left(v - u\frac{dv}{du}\right)^2,$$

or
$$k^2\left(v - u\frac{dv}{du}\right)^2 = 1 - k'^2\left(\frac{dv}{du}\right)^2.$$

This equation is the equivalent, in the new variables, of the equation

$$\frac{dz}{dy} = -1.$$

It is a differential equation of Clairaut's type, and its integral is therefore

$$k^2 (v - uc)^2 = 1 - k'^2 c^2,$$

where c is an arbitrary constant.

Thus the equation

$$k^2 (\operatorname{sn} z \operatorname{sn} y - c \operatorname{cn} z \operatorname{cn} y)^2 = 1 - k'^2 c^2$$

must be equivalent to the equation

$$y + z = x,$$

where c is some function of x.

To determine c in terms of x, put $y = 0$; then we have

$$k^2 c^2 \operatorname{cn}^2 x = 1 - k'^2 c^2,$$

which gives $$c^2 = \operatorname{dn}^{-2} x = \operatorname{dn}^{-2} (z + y).$$

Now the integral equation can be written in the form

$$c^2 (1 - k^2 + k^2 \operatorname{cn}^2 y \operatorname{cn}^2 z) - 2ck^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z + (k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z - 1) = 0.$$

Solving this equation in c, we have

$$c = \frac{k^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z \pm \{k^4 \operatorname{sn}^2 y \operatorname{sn}^2 z \operatorname{cn}^2 y \operatorname{cn}^2 z - (1 - k^2 + k^2 \operatorname{cn}^2 y \operatorname{cn}^2 z)(k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z - 1)\}^{\frac{1}{2}}}{1 - k^2 + k^2 \operatorname{cn}^2 y \operatorname{cn}^2 z},$$

or $$c = \frac{k^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z \pm \operatorname{dn} y \operatorname{dn} z}{1 - k^2 + k^2 \operatorname{cn}^2 y \operatorname{cn}^2 z}.$$

Since

$$k^4 \operatorname{sn}^2 y \operatorname{sn}^2 z \operatorname{cn}^2 y \operatorname{cn}^2 z - \operatorname{dn}^2 y \operatorname{dn}^2 z = (1 - k^2 + k^2 \operatorname{cn}^2 y \operatorname{cn}^2 z)(k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z - 1),$$

this equation can be written

$$c = \frac{k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z - 1}{k^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z \mp \operatorname{dn} y \operatorname{dn} z},$$

or $$\operatorname{dn}(z + y) = \frac{\pm \operatorname{dn} y \operatorname{dn} z \pm k^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z}{1 - k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z}.$$

The two ambiguities of sign in this equation remain to be decided. Taking $z = 0$, it is seen that the first ambiguous sign must be $+$; so

$$\operatorname{dn}(z + y) = \frac{\operatorname{dn} y \operatorname{dn} z \pm k^2 \operatorname{sn} y \operatorname{sn} z \operatorname{cn} y \operatorname{cn} z}{1 - k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z}.$$

Now suppose that y is a small quantity; expanding both sides in ascending powers of y, and retaining only the terms involving the first power of y, we have

$$\operatorname{dn} z + y \frac{d}{dz} \operatorname{dn} z = \operatorname{dn} z \pm k^2 y \operatorname{sn} z \operatorname{cn} z.$$

Since $$\frac{d}{dz} \operatorname{dn} z = - k^2 \operatorname{sn} z \operatorname{cn} z,$$

it is clear that the ambiguous sign must be $-$. We thus finally obtain the *addition-theorem for the function* dn, *namely*

$$\operatorname{dn}(z+y) = \frac{\operatorname{dn} z \operatorname{dn} y - k^2 \operatorname{sn} z \operatorname{sn} y \operatorname{cn} z \operatorname{cn} y}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

Example 1. Shew that

$$\operatorname{dn}(z+y) \operatorname{dn}(z-y) = \frac{\operatorname{dn}^2 y - k^2 \operatorname{cn}^2 y \operatorname{sn}^2 z}{1 - k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z}.$$

Example 2. Prove that

$$1 + \operatorname{dn} 2z = \frac{2 \operatorname{dn}^2 z}{1 - k^2 \operatorname{sn}^4 z}.$$

193. *The addition-theorems for the functions* sn z *and* cn z.

To obtain the addition-theorem for the function sn z, we have

$$\operatorname{sn}(z+y) = \pm \frac{1}{k} \{1 - \operatorname{dn}^2(z+y)\}^{\frac{1}{2}}.$$

Substituting for dn $(z+y)$ from the result of the last article, this equation after some algebraical reduction gives

$$\operatorname{sn}(z+y) = \pm \frac{\operatorname{sn} z \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} z \operatorname{dn} z}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

On putting $y = 0$ in this formula, it is seen that the ambiguous sign is $+$; we thus obtain *the addition-theorem for the function* sn, *namely*

$$\operatorname{sn}(z+y) = \frac{\operatorname{sn} z \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} z \operatorname{dn} z}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

Similarly for the function cn z we obtain the addition-theorem

$$\operatorname{cn}(z+y) = \frac{\operatorname{cn} z \operatorname{cn} y - \operatorname{sn} z \operatorname{dn} z \operatorname{sn} y \operatorname{dn} y}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

These results may be regarded as analogous to the addition-theorems for the circular-functions, namely

$$\sin(z+y) = \sin z \cos y + \cos z \sin y,$$

$$\cos(z+y) = \cos z \cos y - \sin z \sin y,$$

to which, indeed, they reduce when k is put equal to zero.

Example 1. Prove that

$$\operatorname{sn}(z+y) \operatorname{sn}(z-y) = \frac{\operatorname{sn}^2 z - \operatorname{sn}^2 y}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y},$$

$$\operatorname{cn}(z+y) \operatorname{cn}(z-y) = \frac{\operatorname{cn}^2 y - \operatorname{dn}^2 y \operatorname{sn}^2 z}{1 - k^2 \operatorname{sn}^2 y \operatorname{sn}^2 z}.$$

Example 2. Shew that

$$\operatorname{sn}^2 z = \frac{1 - \operatorname{cn} 2z}{1 + \operatorname{dn} 2z}.$$

194. *The constant K.*

We shall denote the integral

$$\int_0^1 (1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}\,dt$$

by K; it is clearly a constant depending only on the modulus k. The ambiguity of sign in the radical will be removed by the supposition that at the lower limit of integration the integrand has the value 1.

From the equation

$$z=\int_0^{\operatorname{sn} z}(1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}\,dt,$$

we see that $$\operatorname{sn} K = 1,$$

and hence $$\operatorname{cn} K = (1-\operatorname{sn}^2 K)^{\frac{1}{2}} = 0,$$

$$\operatorname{dn} K = (1-k^2\operatorname{sn}^2 K)^{\frac{1}{2}} = k'.$$

Example. Prove that

$$\operatorname{sn} \tfrac{1}{2}K = (1+k')^{-\frac{1}{2}},$$

$$\operatorname{cn} \tfrac{1}{2}K = k'^{\frac{1}{2}}(1+k')^{-\frac{1}{2}},$$

$$\operatorname{dn} \tfrac{1}{2}K = k'^{\frac{1}{2}}.$$

195. *The periodicity of the elliptic functions with respect to K.*

It will now appear that the constant K is intimately connected with the periodicity of the elliptic functions sn z, cn z, dn z.

For by the addition-theorem, we have

$$\operatorname{sn}(z+K)=\frac{\operatorname{sn} z\operatorname{cn} K\operatorname{dn} K+\operatorname{sn} K\operatorname{cn} z\operatorname{dn} z}{1-k^2\operatorname{sn}^2 z\operatorname{sn}^2 K}=\frac{\operatorname{cn} z}{\operatorname{dn} z}.$$

Similarly $$\operatorname{cn}(z+K)=-k'\frac{\operatorname{sn} z}{\operatorname{dn} z}$$

and $$\operatorname{dn}(z+K)=\frac{k'}{\operatorname{dn} z}.$$

Hence $$\operatorname{sn}(z+2K)=\frac{\operatorname{cn}(z+K)}{\operatorname{dn}(z+K)}=-\operatorname{sn} z,$$

and similarly $$\operatorname{cn}(z+2K)=-\operatorname{cn} z,$$

$$\operatorname{dn}(z+2K)=\operatorname{dn} z;$$

and finally $$\operatorname{sn}(z+4K)=-\operatorname{sn}(z+2K)=\operatorname{sn} z,$$

$$\operatorname{cn}(z+4K)=\operatorname{cn} z,$$

$$\operatorname{dn}(z+4K)=\operatorname{dn} z.$$

This $4K$ *is a period for the functions* $\operatorname{sn} z$ *and* $\operatorname{cn} z$, *and* $2K$ *is a period for the function* $\operatorname{dn} z$.

Example. If $\operatorname{cs} z = \operatorname{cn} z / \operatorname{sn} z$, shew that

$$\operatorname{cs} z \operatorname{cs}(K-z) = k'.$$

196. *The constant K'.*

We shall denote the integral

$$\int_0^1 (1-t^2)^{-\frac{1}{2}} (1-k'^2 t^2)^{-\frac{1}{2}} dt$$

by K'.

The ambiguity of sign in the radical will be removed by supposing that at the lower limit of the integration the integrand has the value 1.

Write $$s = (1-k'^2 t^2)^{-\frac{1}{2}}.$$

Then $$(1-s^2)^{-\frac{1}{2}} = \frac{i}{k't}(1-k'^2t^2)^{\frac{1}{2}}, \text{ and } (1-k^2s^2)^{-\frac{1}{2}} = \frac{(1-k'^2t^2)^{\frac{1}{2}}}{k'(1-t^2)^{\frac{1}{2}}},$$

and $$ds = (1-k'^2t^2)^{-\frac{3}{2}} k'^2 t\, dt.$$

Therefore $$K' = -i\int_1^{\frac{1}{k}} (1-s^2)^{-\frac{1}{2}} (1-k^2s^2)^{-\frac{1}{2}} ds,$$

and so $$K + iK' = \int_0^{\frac{1}{k}} (1-s^2)^{-\frac{1}{2}} (1-k^2s^2)^{-\frac{1}{2}} ds,$$

or $$\operatorname{sn}(K+iK') = \frac{1}{k},$$

whence $$\operatorname{dn}(K+iK') = 0 \text{ and } \operatorname{cn}(K+iK') = \pm\frac{ik'}{k}.$$

To determine the ambiguous sign in the last equation, we observe that the sign of i must be understood in the light of the relation

$$(1-s^2)^{-\frac{1}{2}} = \frac{i}{k't}(1-k'^2t^2)^{\frac{1}{2}},$$

which was used in the transformation; putting

$$s = \operatorname{sn}(K+iK') = \frac{1}{k}, \quad t = 1,$$

we have $$\frac{1}{\operatorname{cn}(K+iK')} = (1-s^2)^{-\frac{1}{2}} = \frac{ik}{k'},$$

and so $$\operatorname{cn}(K+iK') = -\frac{ik'}{k}.$$

Example. Shew that $$\operatorname{cn}\tfrac{1}{2}(K+iK') = (1-i)\left(\frac{k'}{2k}\right)^{\frac{1}{2}}.$$

197. *The periodicity of the elliptic functions with respect to $K+iK'$.*

The quantity K' introduced in the last article is of importance in connexion with the second period of the functions sn z, cn z, dn z.

For by the addition-theorem, we have

$$\operatorname{sn}(z+K+iK')=\frac{\operatorname{sn} z \operatorname{cn}(K+iK')\operatorname{dn}(K+iK')+\operatorname{sn}(K+iK')\operatorname{cn} z \operatorname{dn} z}{1-k^2\operatorname{sn}^2 z \operatorname{sn}^2(K+iK')}$$

$$=\frac{\operatorname{dn} z}{k\operatorname{cn} z}.$$

Similarly $$\operatorname{cn}(z+K+iK')=-\frac{ik'}{k}\frac{1}{\operatorname{cn} z},$$

and $$\operatorname{dn}(z+K+iK')=\frac{ik'\operatorname{sn} z}{\operatorname{cn} z}.$$

By repeated application of these formulae we have

$$\begin{cases}\operatorname{sn}(z+2K+2iK')=-\operatorname{sn} z,\\ \operatorname{cn}(z+2K+2iK')=\quad\operatorname{cn} z,\\ \operatorname{dn}(z+2K+2iK')=-\operatorname{dn} z,\end{cases}$$

and $$\begin{cases}\operatorname{sn}(z+4K+4iK')=\operatorname{sn} z,\\ \operatorname{cn}(z+4K+4iK')=\operatorname{cn} z,\\ \operatorname{dn}(z+4K+4iK')=\operatorname{dn} z.\end{cases}$$

Hence it appears that *the function* cn z *admits the period* $2K+2iK'$, *and the functions* sn z *and* dn z *admit the period* $4K+4iK'$.

198. *The periodicity of the elliptic functions with respect to iK'.*

By the addition-theorem, we have

$$\operatorname{sn}(z+iK')=\operatorname{sn}(z+K+iK'-K)$$

$$=\frac{\operatorname{sn}(z+K+iK')\operatorname{cn} K\operatorname{dn} K-\operatorname{sn} K\operatorname{cn}(z+K+iK')\operatorname{dn}(z+K+iK')}{1-k^2\operatorname{sn}^2 K\operatorname{sn}^2(z+K+iK')}$$

$$=\frac{1}{k\operatorname{sn} z}.$$

Similarly we find the equations

$$\operatorname{cn}(z+iK')=-\frac{i}{k}\frac{\operatorname{dn} z}{\operatorname{sn} z},$$

$$\operatorname{dn}(z+iK')=-i\frac{\operatorname{cn} z}{\operatorname{sn} z}.$$

By repeated application of these formulae we obtain

$$\begin{cases} \operatorname{sn}(z+2iK') = \operatorname{sn} z, \\ \operatorname{cn}(z+2iK') = -\operatorname{cn} z, \\ \operatorname{dn}(z+2iK') = -\operatorname{dn} z, \end{cases}$$

and

$$\begin{cases} \operatorname{sn}(z+4iK') = \operatorname{sn} z, \\ \operatorname{cn}(z+4iK') = \operatorname{cn} z, \\ \operatorname{dn}(z+4iK') = \operatorname{dn} z, \end{cases}$$

so that *the function* $\operatorname{sn} z$ *admits* $2iK'$ *as a period, and the functions* $\operatorname{cn} z$ *and* $\operatorname{dn} z$ *admit* $4iK'$ *as a period.*

199. *The behaviour of the functions* $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$, *at the point* $z = iK'$.

For points in the neighbourhood of the point $z = 0$, the function $\operatorname{sn} z$ can be expanded by Taylor's theorem in the form

$$\operatorname{sn} z = \operatorname{sn} 0 + z \operatorname{sn}' 0 + \frac{1}{2} z^2 \operatorname{sn}'' 0 + \dots,$$

where accents denote derivatives.

Since

$$\operatorname{sn} 0 = 0,$$

$$\operatorname{sn}' 0 = \operatorname{cn} 0 \operatorname{dn} 0 = 1,$$

$$\operatorname{sn}'' 0 = 0,$$

$$\operatorname{sn}''' 0 = -(1+k^2), \text{ etc.}$$

the expansion becomes

$$\operatorname{sn} z = z - \frac{1}{6}(1+k^2) z^3 + \dots.$$

Hence

$$\operatorname{cn} z = (1 - \operatorname{sn}^2 z)^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2} z^2 + \dots,$$

and

$$\operatorname{dn} z = (1 - k^2 \operatorname{sn}^2 z)^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2} k^2 z^2 + \dots;$$

and therefore

$$\operatorname{sn}(z + iK') = \frac{1}{k \operatorname{sn} z}$$

$$= \frac{1}{kz} \left\{1 - \frac{1}{6}(1+k^2) z^2 + \dots\right\}^{-1}$$

$$= \frac{1}{kz} + \frac{1+k^2}{6k} z + \dots;$$

and similarly
$$\operatorname{cn}(z+iK') = \frac{-i}{kz} + \frac{2k^2-1}{6k}\,iz + \dots$$

and
$$\operatorname{dn}(z+iK') = -\frac{i}{z} + \frac{2-k^2}{6}\,iz + \dots.$$

It follows that *at the point* $z = iK'$, *the functions* sn z, cn z, dn z *have simple poles, with the residues*

$$\frac{1}{k}, \qquad -\frac{i}{k}, \qquad -i,$$

respectively.

200. *General description of the functions* sn z, cn z, dn z.

Summarizing the foregoing investigations, we can describe the functions sn z, cn z, and dn z, in the following terms.

(1) sn z is a one-valued doubly-periodic function of z, its periods being $4K$ and $2iK'$. Its singularities are at all points congruent with $z = iK'$ and $z = 2K + iK'$; they are simple poles, with the residues k^{-1} and $-k^{-1}$ respectively; and the function is zero at all points congruent with $z = 0$ and $z = 2K$.

It may be observed that no other function than sn z exists which fulfils this description. For if $\phi(z)$ be such a function, then

$$\phi(z) - \operatorname{sn} z$$

has no singularities, and so by Liouville's theorem is a constant independent of z; but it is zero when $z = 0$, and therefore the constant is zero; that is,

$$\phi(z) = \operatorname{sn} z.$$

When k^2 is real and positive and less than unity, it is easily seen that K and K' are real, and sn z is real for real values of z and purely imaginary for purely imaginary values of z.

(2) cn z is a one-valued doubly-periodic function of z, its periods being $4K$ and $2K + 2iK'$. Its singularities are at all points congruent with $z = iK'$ and $z = 2K + iK'$; they are simple poles, with the residues ik^{-1} and $-ik^{-1}$ respectively; and the function is zero at all points congruent with $z = K$ and $z = 3K$.

(3) dn z is a one-valued doubly-periodic function of z, its periods being $2K$ and $4iK'$. Its singularities are at all points congruent with $z = iK'$ and $z = 3iK'$; they are simple poles, with the residues $-i$ and $+i$ respectively; and the function is zero at all points congruent with $z = K + iK'$ and $z = K + 3iK'$.

201. *A geometrical illustration of the functions* sn z, cn z, dn z.

The Jacobian elliptic functions may be geometrically represented in the following way.

Let the position of a point, on the surface of a sphere of radius unity, be defined by (1) its perpendicular distance ρ from a fixed diameter of the

sphere, which we shall call the *polar axis*, and (2) the angle ψ which the plane through the point and the polar axis (the *meridian plane*) makes with a fixed plane through the polar axis.

Then if ds denotes the arc of any curve traced on the sphere, we clearly have the relation

$$(ds)^2 = \rho^2 (d\psi)^2 + (1-\rho^2)^{-1} (d\rho)^2.$$

Let a curve (Seiffert's *spherical spiral*) be drawn on the sphere, its defining-equation being

$$d\psi = k\,ds,$$

where k is a constant. We have therefore for this curve

$$(ds)^2 (1 - k^2\rho^2) = (1-\rho^2)^{-1} (d\rho)^2,$$

and so if s be measured from the *pole*, or point where the polar axis meets the sphere, we have

$$s = \int_0^\rho (1-\rho^2)^{-\frac{1}{2}} (1-k^2\rho^2)^{-\frac{1}{2}} d\rho,$$

or

$$\rho = \operatorname{sn} s,$$

the function sn being formed with the modulus k.

The rectangular coordinates of the point s of the curve, referred to the polar axis and an axis perpendicular to it in the meridian-plane, are ρ and $(1-\rho^2)^{\frac{1}{2}}$, and can therefore be written $\operatorname{sn} s$ and $\operatorname{cn} s$; while $\operatorname{dn} s$ is easily seen to be the cosine of the angle at which the curve cuts the meridian. Hence if K be the length of the curve from the pole to the equator, it is obvious that $\operatorname{sn} s$ and $\operatorname{cn} s$ have the period $4K$, and $\operatorname{dn} s$ has the period $2K$.

202. *Connexion of the function* $\operatorname{sn} z$ *with the function* $\wp(z)$.

We shall now shew how the functions considered in this chapter are related to the elliptic function of Weierstrass.

Let e_i, e_j, e_k denote the quantities e_1, e_2, e_3, taken in any order.

In the integral

$$z = \int_{\wp(z)}^\infty \frac{1}{2} (x-e_1)^{-\frac{1}{2}} (x-e_2)^{-\frac{1}{2}} (x-e_3)^{-\frac{1}{2}} dx,$$

let the variable of integration be changed by the substitution

$$x = e_j + \frac{e_i - e_j}{t^2}.$$

Thus

$$z = \int_0^{\left\{\frac{e_i-e_j}{\wp(z)-e_j}\right\}^{\frac{1}{2}}} (1-t^2)^{-\frac{1}{2}} \{(e_i - e_j) + (e_j - e_k) t^2\}^{-\frac{1}{2}} dt,$$

or

$$(e_i - e_j)^{\frac{1}{2}} z = \int_0^{\left\{\frac{e_i-e_j}{\wp(z)-e_j}\right\}^{\frac{1}{2}}} (1-t^2)^{-\frac{1}{2}} (1-k^2t^2)^{-\frac{1}{2}} dt,$$

where
$$k^2 = \frac{e_k - e_j}{e_i - e_j}.$$

This is clearly equivalent to the equation

$$\frac{e_i - e_j}{\wp(z) - e_j} = \text{sn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}.$$

We thus obtain the result that *the function* $\wp(z)$, *formed with any periods, can be expressed in terms of the function* sn z *by the equation*

$$\wp(z) = e_j + \frac{e_i - e_j}{\text{sn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}},$$

the function sn *being formed with the modulus*

$$k = \left(\frac{e_k - e_j}{e_i - e_j}\right)^{\frac{1}{2}}.$$

Example. Shew that this relation can be written in either of the forms

$$\wp(z) = \frac{e_i - e_j \,\text{cn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}}{1 - \text{cn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}},$$

and
$$\wp(z) = \frac{e_k - e_j \,\text{dn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}}{1 - \text{dn}^2 \{(e_i - e_j)^{\frac{1}{2}} z\}}.$$

203. *Expansion of* sn z *as a trigonometric series.*

Since sn z is an odd function of z, admitting the period $4K$ (which we shall for our present purpose suppose to be real), it can by Fourier's theorem be expanded in a series of the form

$$\text{sn}\, z = b_1 \sin \frac{\pi z}{2K} + b_2 \sin \frac{2\pi z}{2K} + b_3 \sin \frac{3\pi z}{2K} + \dots,$$

where (§ 82)
$$b_r = \frac{1}{K} \int_0^{2K} \text{sn}\, t \sin \frac{r\pi t}{2K}\, dt.$$

This expansion will (§ 78) be valid for all points in the z-plane contained in a belt parallel to the real axis and bounded by the lines whose equation is

$$\text{Imaginary part of } z = \pm iK',$$

since within this belt the function sn z has no singularities.

We have now to evaluate the integrals b_r. We shall follow a proof due substantially to Schlömilch.

Let $OARSCBQPO$ be a figure in the plane of a variable t, consisting of the rectangle whose vertices are the points

$$O\,(t=0), \qquad A\,(t = 2K), \qquad C\,(t = 2K + 2iK'), \qquad B\,(t = 2iK'),$$

with a very small semi-circular indentation PQ around the point $t = iK'$, and another small semi-circular indentation RS round the point $t = 2K + iK'$.

Consider the integral

$$\int \operatorname{sn} t\, e^{\frac{ir\pi t}{2K}}\, dt,$$

taken round this contour.

Since the integrand is regular everywhere in the interior of the contour, we have (§ 36)

$$\int_{OA} + \int_{AR} + \int_{RS} + \int_{SC} + \int_{CB} + \int_{BQ} + \int_{QP} + \int_{PO} = 0.$$

Consider first the integral along the semi-circular indentation QP. Writing $t = iK' + Re^{i\theta}$, we have

$$\int_{QP} \operatorname{sn} t\, e^{\frac{ir\pi t}{2K}}\, dt = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \operatorname{sn}(iK' + Re^{i\theta})\, e^{-\frac{r\pi K'}{2K}}\, e^{\frac{ir\pi}{2K} Re^{i\theta}}\, Re^{i\theta}\, id\theta$$

$$= \frac{1}{k} e^{-\frac{r\pi K'}{2K}} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{Re^{\frac{ir\pi}{2K} Re^{i\theta} + i\theta}}{\operatorname{sn}(Re^{i\theta})}\, id\theta, \qquad \text{since } \operatorname{sn}(iK' + z) = \frac{1}{k \operatorname{sn} z}$$

$$= \frac{i}{k} e^{-\frac{r\pi K'}{2K}} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (1 + \text{positive powers of } R)\, d\theta$$

$$= -\frac{\pi i}{k} e^{-\frac{r\pi K'}{2K}}, \qquad \text{when } R \text{ tends to zero.}$$

Similarly we have

$$\int_{RS} \operatorname{sn} t\, e^{\frac{ir\pi t}{2K}}\, dt = (-1)^r \frac{\pi i}{k} e^{-\frac{r\pi K'}{2K}}.$$

Since $\operatorname{sn}(z + 2iK') = \operatorname{sn} z$, we have

$$\int_{CB} = -e^{-\frac{r\pi K'}{K}} \int_{OA},$$

and since $\operatorname{sn}(z + 2K) = -\operatorname{sn} z$, we have

$$\int_{AR} = (-1)^r \int_{PO}, \quad \text{and} \quad \int_{SC} = (-1)^r \int_{BQ}.$$

We thus have

$$(1 - e^{-\frac{r\pi K'}{K}}) \left\{\int_{OA}\right\} - \frac{\pi i}{k} e^{-\frac{r\pi K'}{2K}} \{1 - (-1)^r\} + \{1 + (-1)^r\} \left\{\int_{BQ} + \int_{PO}\right\} = 0.$$

Now equate to zero the imaginary parts of this equation. Since

$$\operatorname{sn} t\, e^{\frac{ir\pi t}{2K}}\, dt$$

is real when t is purely imaginary, there is no imaginary part arising from

$$\int_{BQ} + \int_{PO}.$$

Therefore

$$(1 - e^{-\frac{r\pi K'}{K}}) \int_0^{2K} \operatorname{sn} t \sin \frac{r\pi t}{2K} dt = \frac{\pi}{k} e^{-\frac{r\pi K'}{2K}} \{1 - (-1)^r\}.$$

Writing $$q = e^{-\frac{\pi K'}{K}},$$

this equation gives

$$(1 - q^r) K b_r = \frac{\pi}{k} q^{\frac{r}{2}} \{1 - (-1)^r\},$$

or $$b_r = \frac{2\pi q^{\frac{r}{2}}}{kK(1 - q^r)} \text{ if } r \text{ is odd},$$

and $$b_r = 0 \text{ if } r \text{ is even}.$$

Thus finally we have the expansion of sn z as a trigonometric series,

$$\operatorname{sn} z = \frac{2\pi}{kK} \left(\frac{q^{\frac{1}{2}}}{1 - q} \sin \frac{\pi z}{2K} + \frac{q^{\frac{3}{2}}}{1 - q^3} \sin \frac{3\pi z}{2K} + \frac{q^{\frac{5}{2}}}{1 - q^5} \sin \frac{5\pi z}{2K} + \dots \right).$$

Example. Prove that

$$\operatorname{cn} z = \frac{2\pi}{2K} \left\{ \frac{q^{\frac{1}{2}}}{1+q} \cos \frac{\pi z}{2K} + \frac{q^{\frac{3}{2}}}{1+q^3} \cos \frac{3\pi z}{2K} + \frac{q^{\frac{5}{2}}}{1+q^5} \cos \frac{5\pi z}{2K} + \dots \right\}.$$

Miscellaneous Examples.

1. Shew that

$$z = -\int_0^{\frac{\operatorname{cn} z}{\operatorname{dn} z}} (1 - t^2)^{-\frac{1}{2}} (1 - k^2 t^2)^{-\frac{1}{2}} dt.$$

2. Shew that

$$z = \int_1^{\frac{1}{\operatorname{cn} z}} (t^2 - 1)^{-\frac{1}{2}} (k'^2 t^2 + k^2)^{-\frac{1}{2}} dt.$$

3. Prove that

$$\{1 \pm \operatorname{cn}(z+y)\} \{1 \pm \operatorname{cn}(z-y)\} = \frac{(\operatorname{cn} z \pm \operatorname{cn} y)^2}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

4. Prove that

$$1 + \operatorname{cn}(z+y) \operatorname{cn}(z-y) = \frac{\operatorname{dn}^2 z + \operatorname{dn}^2 y}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

5. Prove that

$$\operatorname{dn}^2 z = \frac{k'^2 + \operatorname{dn} 2z + k^2 \operatorname{cn} 2z}{1 + \operatorname{dn} 2z}.$$

6. Prove that

$$\operatorname{sn}(z-y) \operatorname{dn}(z+y) = \frac{\operatorname{sn} z \operatorname{dn} z \operatorname{cn} y - \operatorname{sn} y \operatorname{dn} y \operatorname{cn} z}{1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 y}.$$

7. Shew that

$$\mathrm{sn}^2 \tfrac{1}{2} z = \frac{\mathrm{dn}\, z - \mathrm{cn}\, z}{k'^2 + \mathrm{dn}\, z - k^2 \,\mathrm{cn}\, z}.$$

8. Shew that

$$\mathrm{sn}\,(z + \tfrac{1}{2}K) = (1+k')^{-\frac{1}{2}} \frac{k' \,\mathrm{sn}\, z + \mathrm{cn}\, z \,\mathrm{dn}\, z}{1 - (1-k')\,\mathrm{sn}^2 z},$$

$$\mathrm{sn}\,(z + \tfrac{1}{2} iK') = k^{-\frac{1}{2}} \frac{(1+k)\,\mathrm{sn}\, z + i \,\mathrm{cn}\, z \,\mathrm{dn}\, z}{1 + k\,\mathrm{sn}^2 z}.$$

9. Prove that

$$\sin\,[\sin^{-1}\{\mathrm{sn}\,(z+y)\} + \sin^{-1}\{\mathrm{sn}\,(z-y)\}] = \frac{2\,\mathrm{sn}\, z \,\mathrm{cn}\, z \,\mathrm{dn}\, y}{1 - k^2 \,\mathrm{sn}^2 z \,\mathrm{sn}^2 y}.$$

10. Shew that

$$\cos\,[\sin^{-1}\{\mathrm{sn}\,(z+y)\} - \sin^{-1}\{\mathrm{sn}\,(z-y)\}] = \frac{\mathrm{cn}^2 y - \mathrm{sn}^2 y \,\mathrm{dn}^2 z}{1 - k^2 \,\mathrm{sn}^2 z \,\mathrm{sn}^2 y}.$$

11. Shew that the quarter-periods K and iK' are solutions of the equation

$$z(1-z)\frac{d^2u}{dz^2} + (1-2z)\frac{du}{dz} - \frac{1}{4}u = 0,$$

where $z = k^2$.

12. Shew that the quarter-periods K and iK' are Legendre functions of the argument $(1-2k^2)$, of order $-\frac{1}{2}$.

13. Shew that

$$\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{dn}\,\alpha + \mathrm{cn}\,\gamma\,\mathrm{cn}\,\alpha\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{dn}\,\beta + \mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{sn}\,(\alpha-\beta)\,\mathrm{dn}\,\gamma$$
$$+ \mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma = 0.$$

(Cambridge Mathematical Tripos, Part I, 1894.)

14. If $u+v+r+s=0$, shew that

$$k^2 \,\mathrm{sn}\, u \,\mathrm{sn}\, v \,\mathrm{cn}\, r \,\mathrm{cn}\, s - k^2 \,\mathrm{cn}\, u \,\mathrm{cn}\, v \,\mathrm{sn}\, r \,\mathrm{sn}\, s - \mathrm{dn}\, u \,\mathrm{dn}\, v + \mathrm{dn}\, r \,\mathrm{dn}\, s = 0,$$

$$k'^2 \,\mathrm{sn}\, u \,\mathrm{sn}\, v - k'^2 \,\mathrm{sn}\, r \,\mathrm{sn}\, s + \mathrm{dn}\, u \,\mathrm{dn}\, v \,\mathrm{cn}\, r \,\mathrm{cn}\, s - \mathrm{cn}\, u \,\mathrm{cn}\, v \,\mathrm{dn}\, r \,\mathrm{dn}\, s = 0,$$

$$\mathrm{sn}\, u \,\mathrm{sn}\, v \,\mathrm{dn}\, r \,\mathrm{dn}\, s - \mathrm{dn}\, u \,\mathrm{dn}\, v \,\mathrm{sn}\, r \,\mathrm{sn}\, s + \mathrm{cn}\, r \,\mathrm{cn}\, s - \mathrm{cn}\, u \,\mathrm{cn}\, v = 0.$$

(H. J. S. Smith.)

15. Shew that, if $\alpha > x > \beta > \gamma$, the substitutions

$$x - \gamma = (\alpha-\gamma)\,\mathrm{dn}^2 u \quad \text{and} \quad x - \gamma = (\beta-\gamma)\,\mathrm{dn}^{-2} v,$$

where $k^2 = (\alpha-\beta)(\alpha-\gamma)^{-1}$, reduce the integrals

$$\int_x^\alpha \{(\alpha-x)(x-\beta)(x-\gamma)\}^{-\frac{1}{2}}\,dx \quad \text{and} \quad \int_\beta^x \{(\alpha-x)(x-\beta)(x-\gamma)\}^{-\frac{1}{2}}\,dx$$

respectively to the forms $2u(\alpha-\gamma)^{-\frac{1}{2}}$ and $2v(\alpha-\gamma)^{-\frac{1}{2}}$; and deduce that, if $u+v=K$,

$$1 - \mathrm{sn}^2 u - \mathrm{sn}^2 v + k^2 \,\mathrm{sn}^2 u \,\mathrm{sn}^2 v = 0.$$

From the substitution $y = (\alpha-x)(x-\beta)(x-\gamma)^{-1}$, applied to the integral above with the limits β and α, obtain the result

$$\int_0^{\frac{\pi}{2}} (a_1 \cos^2\theta + b_1 \sin^2\theta)^{-\frac{1}{2}}\,d\theta = \int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta,$$

where a_1, b_1 are the arithmetic and geometric means between a and b.

(Cambridge Mathematical Tripos, Part I, 1895.)

16. Shew how to express

$$\int \{(ax^2+bx+c)(a'x^2+b'x+c')\}^{-\frac{1}{2}}\,dx$$

as an elliptic integral of the first kind, in the case when both quadratic expressions have imaginary linear factors.

If

$$z=\int_x^\infty \{(x+1)(x^2+x+1)\}^{-\frac{1}{2}}\,dx,$$

express x in terms of z by means of Jacobi's elliptic functions.

(Cambridge Mathematical Tripos, Part I, 1899.)

17. The different values of z satisfying the equation $\operatorname{cn} 3z=a$ are $z_1, z_2, \ldots z_9$. Shew that

$$3k^4 \prod_{r=1}^{9} \operatorname{cn} z_r + k'^4 \prod_{r=9}^{9} \operatorname{cn} z_r = 0.$$

(Cambridge Mathematical Tripos, Part I, 1899.)

18. Shew that

$$\frac{\operatorname{cn} z}{\operatorname{dn} z}=\frac{2\pi}{kK}\left\{\frac{q^{\frac{1}{2}}}{1-q}\cos\frac{\pi z}{2K}-\frac{q^{\frac{3}{2}}}{1-q^3}\cos\frac{3\pi z}{2K}+\frac{q^{\frac{5}{2}}}{1-q^5}\cos\frac{5\pi z}{2K}-\ldots\right\}.$$

19. Prove that

$$\frac{k'\operatorname{sn} z}{\operatorname{dn} z}=\frac{2\pi}{kK}\left\{\frac{q^{\frac{1}{2}}}{1+q}\sin\frac{\pi z}{2K}-\frac{q^{\frac{3}{2}}}{1+q^3}\sin\frac{3\pi z}{2K}+\frac{q^{\frac{5}{2}}}{1+q^5}\sin\frac{5\pi z}{2K}-\ldots\right\}.$$

20. Shew that

$$\operatorname{dn} z=\frac{\pi}{2K}+\frac{2\pi}{K}\left\{\frac{q}{1+q^2}\cos\frac{\pi z}{K}+\frac{q^2}{1+q^4}\cos\frac{2\pi z}{K}+\ldots\right\}.$$

21. Prove that

$$\begin{aligned}\operatorname{sn}^3 z=&\left\{\frac{1+k^2}{2k^3}\left(\frac{\pi}{2K}\right)-\frac{1^2}{2k^3}\left(\frac{\pi}{2K}\right)^3\right\}\frac{4q^{\frac{1}{2}}}{1-q}\sin\frac{\pi z}{2K}\\ &+\left\{\frac{1+k^2}{2k^3}\left(\frac{\pi}{2K}\right)-\frac{3^2}{2k^3}\left(\frac{\pi}{2K}\right)^3\right\}\frac{4q^{\frac{3}{2}}}{1-q^3}\sin\frac{3\pi z}{2K}\\ &+\left\{\frac{1+k^2}{2k^3}\left(\frac{\pi}{2K}\right)-\frac{5^2}{2k^3}\left(\frac{\pi}{2K}\right)^3\right\}\frac{4q^{\frac{5}{2}}}{1-q^5}\sin\frac{5\pi z}{2K}\\ &+\ldots.\end{aligned}$$

(Cambridge Mathematical Tripos, Part II, 1896.)

22. Shew that

$$k^2\operatorname{sn}^2 z=\wp(z-iK')+\text{Constant},$$

where the Weierstrassian elliptic function is formed with the periods $2K$ and $2iK'$.

23. Shew that the differential equation

$$\frac{d^2u}{dz^2}=\{\tfrac{3}{4}k^2\operatorname{sn}^2 z-\tfrac{1}{4}(1+k^2)\}\,u$$

admits the general integral

$$u=\{\operatorname{sn}\tfrac{1}{2}(C-z)\operatorname{cn}\tfrac{1}{2}(C-z)\operatorname{dn}\tfrac{1}{2}(C-z)\}^{-\frac{1}{2}}\{A+B\operatorname{sn}^2\tfrac{1}{2}(C-z)\},$$

where A and B are arbitrary constants, and $C=2K+iK'$.

CHAPTER XVI.

Elliptic Functions; General Theorems.

204. *Relation between the residues of an elliptic function.*

In this chapter we shall be chiefly concerned with properties of more general elliptic functions than the special functions $\wp(z)$, $\operatorname{sn} z$, $\operatorname{cn} z$, and $\operatorname{dn} z$, which have been discussed in the two preceding chapters.

We shall first shew that *the sum of the residues of any elliptic function, with respect to those of its poles which are situated in any period-parallelogram, is zero.*

For let $f(z)$ be an elliptic function, and let $2\omega_1$ and $2\omega_2$ be its periods. The sum of the residues is, by § 56, equal to the integral

$$\frac{1}{2\pi i}\int f(z)\,dz$$

taken round the perimeter of the parallelogram.

Now in this integral, any two elements $f(z)\,dz$ corresponding to congruent line-elements dz on opposite sides of the parallelogram, are equal in magnitude but opposite in sign, and therefore destroy each other. Hence the integral is zero, which establishes the theorem.

The number of poles or zeros of an elliptic function contained within a single period-parallelogram is often referred to as the number of *irreducible* poles or zeros.

205. *The order of an elliptic function.*

We shall next shew that if c is any constant and $f(z)$ is an elliptic function, *the number of roots of the equation*

$$f(z) = c$$

contained within a period-parallelogram depends only on $f(z)$, and is independent of c, and is therefore equal to the number of irreducible zeros, and also to the number of irreducible poles.

For the difference between the number of zeros of the function

$$f(z)-c$$

and the number of its poles, contained within the parallelogram, is (§ 60) equal to the value of the integral

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)-c}\,dz$$

taken round the perimeter of the parallelogram. But if P and Q are two points congruent with each other, situated on opposite sides of the parallelogram, then the elements $f'(z)\{f(z)-c\}^{-1}dz$ arising from P and Q are equal in magnitude but opposite in sign, and so destroy each other. The integral is therefore zero; that is, the number of zeros of the function $f(z)-c$ contained within the parallelogram is equal to the number of its poles, i.e. to the number of the poles of $f(z)$; but this latter number is independent of c, which establishes the theorem.

The number of irreducible poles or zeros of an elliptic function is called the *order* of the function. It must be noted that a zero or pole, which is multiple of order n in the sense of "order" defined in §§ 39, 44, must be counted as n zeros or poles for the purposes of this definition of "order."

The order is never less than two; for if an elliptic function had only a single irreducible simple pole, the sum of its residues within any period-parallelogram would not be zero, contrary to the theorem of the last article. This explains why the functions discussed in the two preceding chapters, which are of order two, are the *simplest* elliptic functions.

206. *Expression of any elliptic function in terms of* $\wp(z)$ *and* $\wp'(z)$.

We shall now shew how any elliptic function can be expressed in terms of the Weierstrassian elliptic function which has the same periods.

Let $f(z)$ be any elliptic function, and let $\wp(z)$ be the Weierstrassian elliptic function with the same periods $2\omega_1$ and $2\omega_2$; and let $\wp'(z)$ be the derivate of $\wp(z)$.

First, we can write

$$f(z)=\frac{1}{2}\{f(z)+f(-z)\}+\frac{1}{2}\frac{f(z)-f(-z)}{\wp'(z)}\wp'(z).$$

Now the functions $f(z)+f(-z)$ and $\{f(z)-f(-z)\}\wp'^{-1}(z)$ are *even* elliptic functions of z: let $\phi(z)$ denote either of them: we shall now express $\phi(z)$ in terms of $\wp(z)$.

Since $\phi(z)$ is an even function, it follows that if a be one of its zeros in the fundamental period-parallelogram, then another of its zeros in the parallelogram will be congruent to $-a$: its irreducible zeros may therefore be arranged in two sets, say $a_1, a_2, \dots a_n$, and zeros congruent to

$$-a_1, -a_2, \dots -a_n.$$

Similarly its poles can be arranged in two sets, say $b_1, b_2, \ldots b_n$, and poles congruent to $-b_1, -b_2, \ldots -b_n$.

Now form the quantity

$$\frac{1}{\phi(z)} \frac{\{\wp(z)-\wp(a_1)\}\{\wp(z)-\wp(a_2)\}\ldots\{\wp(z)-\wp(a_n)\}}{\{\wp(z)-\wp(b_1)\}\{\wp(z)-\wp(b_2)\}\ldots\{\wp(z)-\wp(b_n)\}},$$

where if one of the quantities a_r or b_r is zero, the corresponding factor $\{\wp(z)-\wp(a_r)\}$ or $\{\wp(z)-\wp(b_r)\}$ is to be omitted.

This quantity is a doubly-periodic function of z; it clearly has no zeros or singularities in the interior of the parallelogram, except possibly at $z=0$, and therefore either it or its reciprocal has no singularities in the interior of the parallelogram, and so has no singularities in the entire plane. It must therefore by Liouville's theorem be a constant independent of z.

Thus

$$\phi(z) = \text{Constant} \times \frac{\{\wp(z)-\wp(a_1)\}\{\wp(z)-\wp(a_2)\}\ldots\{\wp(z)-\wp(a_n)\}}{\{\wp(z)-\wp(b_1)\}\{\wp(z)-\wp(b_2)\}\ldots\{\wp(z)-\wp(b_n)\}}.$$

The quantities $\{f(z)+f(-z)\}$ and $\{f(z)-f(-z)\}\{\wp'(z)\}^{-1}$ can thus be expressed as rational functions of $\wp(z)$; and thus we obtain the theorem that *any elliptic function can be expressed in terms of the Weierstrassian function formed with the same periods, the expression being linear in* $\wp'(z)$ *and rational in* $\wp(z)$.

Example. Shew that

$$\operatorname{sn} z \operatorname{cn} z \operatorname{dn} z = \tfrac{1}{2} k^{-2} \wp'(z-iK'),$$

where the function $\wp'(z-iK')$ is formed with the periods $2K$ and $2iK'$.

207. *Relation between any two elliptic functions which admit the same periods.*

We shall now shew that *an algebraic relation exists between any two elliptic functions whose periods are the same.*

For let $f(z)$ and $\phi(z)$ be the functions. Then by the last article, $f(z)$ and $\phi(z)$ can be expressed rationally in terms of $\wp(z)$ and $\wp'(z)$. Eliminating $\wp(z)$ and $\wp'(z)$ from the three equations constituted by

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$$

and these two relations, we have an algebraic relation between $f(z)$ and $\phi(z)$; which establishes the theorem.

It is easy to find the degree of this equation in f and ϕ. For if f be an elliptic function of order m, and if ϕ be of order n, then each value of f determines m irreducible values of z, and each of these determines one value

of ϕ: so to each value of f correspond m values of ϕ, and similarly to each value of ϕ correspond n values of f. The equation is therefore of degree m in ϕ and n in f.

Thus $\wp(z)$ is of order 2, and $\wp'(z)$ of order 3. The relation between them, namely

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3,$$

should therefore be of degree 2 in $\wp'(z)$ and 3 in $\wp(z)$—as in fact it is.

An obvious consequence of this proposition is that *every elliptic function is connected with its derivate by an algebraic equation.*

Example. If t, u, v are three elliptic functions of the second order, with the same periods and argument, shew that there exist in general between them two distinct relations which are linear with respect to each of them, namely

$$Atuv + Buv + Cvt + Dtu + Et + Fu + Gv + H = 0,$$

$$A'tuv + B'uv + C'vt + D'tu + E't + F'u + G'v + H' = 0,$$

where $A, B, \ldots, H'$ are constants.

208. *Relation between the zeros and poles of an elliptic function.*

We shall now shew that *the sum of the affixes of the irreducible zeros of an elliptic function is equal to the sum of the affixes of its irreducible poles, or differs from this sum only by a period.*

For if $f(z)$ be the function, and $2\omega_1$ and $2\omega_2$ its periods, the difference in question is (§ 59) equal to the integral

$$\frac{1}{2\pi i}\int \frac{zf'(z)\,dz}{f(z)}$$

taken round the perimeter of the fundamental period-parallelogram. This can be written

$$\frac{1}{2\pi i}\left\{\int_0^{2\omega_1} + \int_{2\omega_1}^{2\omega_1+2\omega_2} + \int_{2\omega_1+2\omega_2}^{2\omega_2} + \int_{2\omega_2}^{0}\right\}\frac{zf'(z)}{f(z)}\,dz,$$

or

$$\frac{1}{2\pi i}\left[\int_0^{2\omega_1}\left\{\frac{zf'(z)}{f(z)} - \frac{(2\omega_2+z)f'(2\omega_2+z)}{f(2\omega_2+z)}\right\}dz \right.$$
$$\left. + \int_0^{2\omega_2}\left\{-\frac{zf'(z)}{f(z)} + \frac{(2\omega_1+z)f'(2\omega_1+z)}{f(2\omega_1+z)}\right\}dz\right],$$

or

$$\frac{1}{2\pi i}\left\{-2\omega_2\int_0^{2\omega_1}\frac{f'(z)}{f(z)}\,dz + 2\omega_1\int_0^{2\omega_2}\frac{f'(z)}{f(z)}\,dz\right\},$$

or

$$\frac{1}{2\pi i}\left\{-2\omega_2\log\frac{f(2\omega_1)}{f(0)} + 2\omega_1\log\frac{f(2\omega_2)}{f(0)}\right\},$$

or

$$\frac{1}{2\pi i}\{-2\omega_2\log 1 + 2\omega_1\log 1\},$$

and as log 1 is zero or some multiple of $2\pi i$, the last expression must be either zero or some quantity of the form

$$\text{A multiple of } 2\omega_1 + \text{A multiple of } 2\omega_2,$$

i.e. a period. This establishes the theorem.

Example. If $F(z)$ is an elliptic function, for which $z_1, z_2, \ldots$ are the irreducible poles, and $A_1, A_2, \ldots$ the corresponding residues, and if $f(z)$ is a one-valued function without singularities in the parallelogram, shew that the integral

$$\frac{1}{2\pi i}\int f(z)\, F(z)\, dz,$$

taken round the period-parallelogram, is equal to $\Sigma A_n f(z_n)$.

(Cambridge Mathematical Tripos, Part II, 1899.)

209. *The function* $\zeta(z)$.

We shall next introduce a function $\zeta(z)$, defined by the equation

$$\frac{d\zeta(z)}{dz} = -\wp(z),$$

with the condition that $\zeta(z) - z^{-1}$ is to be zero when $z = 0$.

Since the infinite series which represents $\wp(z)$ is uniformly convergent, it can be integrated term by term; we thus have

$$\zeta(z) = -\int [z^{-2} + \Sigma\{(z - 2m\omega_1 - 2n\omega_2)^{-2} - (2m\omega_1 + 2n\omega_2)^{-2}\}]\, dz$$

$$= z^{-1} + \Sigma\{(z - 2m\omega_1 - 2n\omega_2)^{-1} + (2m\omega_1 + 2n\omega_2)^{-1} + z(2m\omega_1 + 2n\omega_2)^{-2}\},$$

since the condition, which $\zeta(z)$ has to satisfy at $z = 0$, is satisfied by this choice of the constant of integration. The summation is, as usual, extended over all positive and negative integer and zero values of m and n, except simultaneous zero values.

When $|2m\omega_1 + 2n\omega_2|$ is large (and we can suppose the series arranged in ascending order of magnitude of $|2m\omega_1 + 2n\omega_2|$), the quantity

$$(z - 2m\omega_1 - 2n\omega_2)^{-1} + (2m\omega_1 + 2n\omega_2)^{-1} + z(2m\omega_1 + 2n\omega_2)^{-2}$$

bears a ratio of approximate equality to the quantity

$$-z^2(2m\omega_1 + 2n\omega_2)^{-3}.$$

The series which represents $\zeta(z)$ can therefore be compared with the series $\Sigma(2m\omega_1 + 2n\omega_2)^{-3}$, and hence we see that it is absolutely convergent, except at the singularities $z = 2m\omega_1 + 2n\omega_2$, and that the convergence is uniform.

It is evident from the series that at its singularities $z = 2m\omega_1 + 2n\omega_2$, the function $\zeta(z)$ has simple poles with residues unity; and that $\zeta(z)$ is an odd function of z.

The function $\zeta(z)$ may be compared with the function cot z, whose expansion is

$$\cot z = z^{-1} + \sum_{m=-\infty}^{\infty} \{(z - m\pi)^{-1} + (m\pi)^{-1}\}.$$

The equation
$$\frac{d}{dz}\cot z = -\operatorname{cosec}^2 z$$
corresponds to the equation
$$\frac{d}{dz}\zeta(z) = -\wp(z).$$

210. *The quasi-periodicity of the function $\zeta(z)$.*

Since
$$\wp(z + 2\omega_1) = \wp(z),$$
we have
$$\frac{d}{dz}\zeta(z + 2\omega_1) = \frac{d}{dz}\zeta(z),$$
or
$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1,$$
and similarly
$$\zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2,$$
where η_1 and η_2 are two constants introduced by integration.

Writing $z = -\omega_1$ and $z = -\omega_2$ in these relations respectively, we have

$$\zeta(\omega_1) = \zeta(-\omega_1) + 2\eta_1 = -\zeta(\omega_1) + 2\eta_1,$$
$$\zeta(\omega_2) = \zeta(-\omega_2) + 2\eta_2 = -\zeta(\omega_2) + 2\eta_2,$$
whence
$$\eta_1 = \zeta(\omega_1),$$
$$\eta_2 = \zeta(\omega_2),$$
which determines the constants η_1 and η_2.

If $x+y+z=0$, shew that

$$\{\zeta(x)+\zeta(y)+\zeta(z)\}^2+\zeta'(x)+\zeta'(y)+\zeta'(z)=0.$$

(Schottky.)

This result may be regarded as the addition-theorem for the function $\zeta(z)$.

211. *Expression of an elliptic function, when the principal part of its expansion at each of its singularities is given.*

Let $f(z)$ be any elliptic function, with periods $2\omega_1$ and $2\omega_2$. Let its irreducible singularities be at the points $z = a_1, a_2, \ldots a_n$; and let the principal part of its expansion near the point a_k be

$$\frac{c_{k_1}}{z - a_k} + \frac{c_{k_2}}{(z - a_k)^2} + \ldots + \frac{c_{k_{r_k}}}{(z - a_k)^{r_k}}.$$

Then if we consider the function

$$E(z) = \sum_{k=1}^{n} \left\{c_{k_1}\zeta(z - a_k) - c_{k_2}\zeta'(z - a_k) + \ldots + \frac{(-1)^{r_k-1}}{(r_k - 1)!}c_{k_{r_k}}\zeta^{(r_k-1)}(z - a_k)\right\},$$

where $\zeta^{(s)}(z)$ denotes $\dfrac{d^s}{dz^s}\zeta(z)$, we see that

(1) When z is replaced by $(z+2\omega_1)$, the function $E(z)$ is increased by

$$2\eta_1 \sum_{k=1}^{n} c_{k_1}.$$

But $\sum_{k=1}^{n} c_{k_1}$ is zero, since the sum of the residues of $f(z)$ within a period-parallelogram is zero. Hence $E(z)$ admits the period $2\omega_1$. Similarly $E(z)$ admits the period $2\omega_2$. $E(z)$ is therefore an elliptic function, with the same periods as $f(z)$.

(2) Since the function $\zeta^{(m)}(z-a_k)$ has singularities only at a_k and congruent points, and its principal part at a_k is $(-1)^m m!(z-a_k)^{-m-1}$, we see that $E(z)$ has the same singularities as $f(z)$, and the same principal parts at them.

It follows from (1) and (2) that $f(z)-E(z)$ is a function with no singularities in the whole plane; and therefore, by Liouville's theorem, $f(z)-E(z)$ is a constant. Thus *the function $f(z)$ can be expanded in the form*

$$f(z) = \text{Constant} + \sum_{k=1}^{n} \sum_{s=1}^{r_k} \frac{(-1)^{s-1}}{(s-1)!} c_{k_s} \zeta^{(s-1)}(z-a_k).$$

This theorem may be regarded as analogous to the decomposition of a rational function into partial fractions, or the decomposition of a circular function into a series of cotangents (§ 76).

Example 1. Shew that

$$k \operatorname{sn} z = \zeta(z-iK') - \zeta(z-2K-iK') + \text{Constant},$$

where the ζ-functions are formed with the periods $4K$ and $2iK'$.

Example 2. Shew that

$$2 \begin{vmatrix} 1 & \wp(x) & \wp^2(x) \\ 1 & \wp(y) & \wp^2(y) \\ 1 & \wp(z) & \wp^2(z) \end{vmatrix} \div \begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(z) & \wp'(z) \end{vmatrix}$$
$$= \zeta(x+y+z) - \zeta(x) - \zeta(y) - \zeta(z).$$

Extend this theorem to the case in which there are any number of variables.

(Cambridge Mathematical Tripos, Part II, 1894.)

212. *The function $\sigma(z)$.*

We shall next introduce a function $\sigma(z)$, defined by the equation

$$\frac{d}{dz} \log \sigma(z) = \zeta(z),$$

with the condition that $\sigma(z)/z$ is to be unity when $z=0$.

Since the convergence of the infinite series which represents $\zeta(z)$ is uniform, the series can be integrated term by term: we thus have

$$\log \sigma(z) = \log z + \Sigma \left\{ \log\left(1 - \frac{z}{2m\omega_1 + 2n\omega_2}\right) + \frac{z}{2m\omega_1 + 2n\omega_2} + \frac{1}{2}\frac{z^2}{(2m\omega_1 + 2n\omega_2)^2}\right\},$$

on choosing the constant of integration so as to satisfy the condition at $z = 0$; and therefore

$$\sigma(z) = z\,\Pi\left(1 - \frac{z}{2m\omega_1 + 2n\omega_2}\right) e^{\frac{z}{2m\omega_1+2n\omega_2} + \frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}},$$

the product being, as usual, extended over all integer and zero values of m and n, except simultaneous zeros. The absolute convergence of this product follows from that of the series

$$\Sigma \left\{ \log\left(1 - \frac{z}{2m\omega_1 + 2n\omega_2}\right) + \frac{z}{2m\omega_1 + 2n\omega_2} + \frac{1}{2}\frac{z^2}{(2m\omega_1 + 2n\omega_2)^2}\right\},$$

which is established by comparison with the series

$$-\Sigma \frac{z^3}{3(2m\omega_1 + 2n\omega_2)^3},$$

since the terms of the two series have ultimately a ratio of equality.

It is evident from the product-expression that $\sigma(z)$ is an odd function of z, that its zeros are at the points $z = 2m\omega_1 + 2n\omega_2$, and that $z^{-1}\sigma(z)$ tends to the limit unity as z tends to zero.

The function $\sigma(z)$ may be compared with the function $\sin z$, defined by the expansion

$$\sin z = z \prod_{m=-\infty}^{\infty} \left\{\left(1 - \frac{z}{m\pi}\right) e^{\frac{z}{m\pi}}\right\}.$$

The relation $$\frac{d}{dz}\log(\sin z) = \cot z$$

corresponds to $$\frac{d}{dz}\log\sigma(z) = \zeta(z).$$

213. *The quasi-periodicity of the function* $\sigma(z)$.

On integrating the equation

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1$$

we have $$\log\sigma(z + 2\omega_1) = \log\sigma(z) + 2\eta_1 z + \text{Constant},$$

or $$\sigma(z + 2\omega_1) = ce^{2\eta_1 z}\sigma(z),$$

where c is a constant. To determine c, write $z = -\omega_1$; thus

$$\sigma(\omega_1) = -ce^{-2\eta_1\omega_1}\sigma(\omega_1),$$

or $$c = -e^{2\eta_1\omega_1},$$

and therefore $$\sigma(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z).$$

Similarly $$\sigma(z+2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z).$$

The behaviour of the function $\sigma(z)$ when its argument is increased by a period of $\wp(z)$ is thus determined. By repeated application of these formulae we can find the value of $\sigma(z+2m\omega_1+2n\omega_2)$, where m and n are any integers.

An example shewing how the function $\sigma(z)$ may be expressed as a singly-infinite product.

We have

$$\sigma(z) = z\Pi\left(1-\frac{z}{2m\omega_1+2n\omega_2}\right)e^{\frac{z}{2m\omega_1+2n\omega_2}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}},$$

the summation being extended over all positive and negative integer and zero values of m and n, except simultaneous zeros. This can be written in the form

$$\sigma(z) = z\prod_{m=\pm1}^{\pm\infty}\left(1-\frac{z}{2m\omega_1}\right)e^{\frac{z}{2m\omega_1}+\frac{1}{2}\frac{z^2}{(2m\omega_1)^2}}\times\prod_{n=\pm1}^{\pm\infty}\left(1-\frac{z}{2n\omega_2}\right)e^{\frac{z}{2n\omega_2}+\frac{1}{2}\frac{z^2}{(2n\omega_2)^2}}$$

$$\times\prod_{m=\pm1}^{\pm\infty}\prod_{n=1}^{\infty}\left(1-\frac{z}{2m\omega_1+2n\omega_2}\right)e^{\frac{z}{2m\omega_1+2n\omega_2}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}}$$

$$\times\prod_{m=\pm1}^{\pm\infty}\prod_{n=1}^{\infty}\left(1+\frac{z}{2m\omega_1+2n\omega_2}\right)e^{\frac{-z}{2m\omega_1+2n\omega_2}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}}.$$

Now

$$z\prod_{m=\pm1}^{\pm\infty}\left(1-\frac{z}{2m\omega_1}\right)e^{\frac{z}{2m\omega_1}+\frac{1}{2}\frac{z^2}{(2m\omega_1)^2}} = \frac{2\omega_1}{\pi}e^{\frac{1}{2}z^2\sum\limits_{m=\pm1}^{\pm\infty}\frac{1}{(2m\omega_1)^2}}\sin\frac{z\pi}{2\omega_1},$$

and

$$\prod_{m=\pm1}^{\pm\infty}\prod_{n=1}^{\infty}\left(1-\frac{z}{2m\omega_1+2n\omega_2}\right)e^{\frac{z}{2m\omega_1+2n\omega_2}+\frac{1}{2}\left(\frac{z}{2m\omega_1+2n\omega_2}\right)^2}$$

$$=\prod_{m=\pm1}^{\pm\infty}\prod_{n=1}^{\infty}\frac{\left(1+\frac{2n\omega_2-z}{2m\omega_1}\right)e^{-\frac{2n\omega_2-z}{2m\omega_1}}}{\left(1+\frac{2n\omega_2}{2m\omega_1}\right)e^{-\frac{2n\omega_2}{2m\omega_1}}}e^{\frac{-2n\omega_2 z}{2m\omega_1(2m\omega_1+2n\omega_2)}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}}$$

$$=\prod_{n=1}^{\infty}\frac{\sin\frac{(2n\omega_2-z)\pi}{2\omega_1}}{\sin\frac{2n\omega_2\pi}{2\omega_1}}\,\frac{1}{1-\frac{z}{2n\omega_2}}\,.\,e^{\sum\limits_{m=\pm1}^{\pm\infty}\left\{\frac{-2n\omega_2 z}{2m\omega_1(2m\omega_1+2n\omega_2)}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}\right\}}.$$

Similarly

$$\prod_{m=\pm1}^{\pm\infty}\prod_{n=1}^{\infty}\left(1+\frac{z}{2m\omega_1+2n\omega_2}\right)e^{\frac{-z}{2m\omega_1+2n\omega_2}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}}$$

$$=\prod_{n=1}^{\infty}\frac{\sin\frac{(2n\omega_2+z)\pi}{2\omega_1}}{\sin\frac{2n\omega_2\pi}{2\omega_1}}\,\frac{1}{1+\frac{z}{2n\omega_2}}\,e^{\sum\limits_{m=\pm1}^{\pm\infty}\left\{\frac{2n\omega_2 z}{2m\omega_1(2m\omega_1+2n\omega_2)}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}\right\}}.$$

Therefore

$$\sigma(z)=\frac{2\omega_1}{\pi}\, e^{\frac{1}{2}z^2 \sum\limits_{m=\pm 1}^{\pm\infty}\frac{1}{(2m\omega_1)^2}} \sin\frac{\pi z}{2\omega_1}$$

$$\times \prod_{n=1}^{\infty}\left[e^{\frac{z}{2n\omega_2}+\frac{1}{2}\frac{z^2}{(2n\omega_2)^2}}\, \frac{\sin\dfrac{(2n\omega_2-z)\pi}{2\omega_1}}{\sin\dfrac{2n\omega_2\pi}{2\omega_1}}\, e^{\sum\limits_{m=\pm 1}^{\pm\infty}\left\{\frac{-2n\omega_2 z}{2m\omega_1(2m\omega_1+2n\omega_2)}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}\right\}}\right]$$

$$\times \prod_{n=1}^{\infty}\left[e^{\frac{-z}{2n\omega_2}+\frac{1}{2}\frac{z^2}{(2n\omega_2)^2}}\, \frac{\sin\dfrac{(2n\omega_2+z)\pi}{2\omega_1}}{\sin\dfrac{2n\omega_2\pi}{2\omega_1}}\, e^{\sum\limits_{m=\pm 1}^{\pm\infty}\left\{\frac{2n\omega_2 z}{2m\omega_1(2m\omega_1+2n\omega_2)}+\frac{1}{2}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}\right\}}\right],$$

or

$$\sigma(z)=\frac{2\omega_1}{\pi}\, e^{\frac{1}{2}z^2 \sum\limits_{m=\pm 1}^{\pm\infty}\frac{1}{(2m\omega_1)^2}} \sin\frac{z\pi}{2\omega_1} \prod_{n=1}^{\infty}\left[\frac{\sin\dfrac{(2n\omega_2-z)\pi}{2\omega_1}\sin\dfrac{(2n\omega_2+z)\pi}{2\omega_1}}{\sin^2\dfrac{n\omega_2\pi}{\omega_1}}\, e^{\frac{z^2}{(2n\omega_2)^2}+\sum\limits_{m=\pm 1}^{\pm\infty}\frac{z^2}{(2m\omega_1+2n\omega_2)^2}}\right].$$

Now write $q=e^{\frac{i\pi\omega_2}{\omega_1}}$.

Then

$$\frac{\sin\dfrac{(2n\omega_2-z)\pi}{2\omega_1}\sin\dfrac{(2n\omega_2+z)\pi}{2\omega_1}}{\sin^2\dfrac{n\omega_2\pi}{\omega_1}}=\frac{\left\{1-e^{(2n\omega_2-z)\frac{i\pi}{\omega_1}}\right\}\left\{1-e^{\frac{i\pi}{\omega_1}(2n\omega_2+z)}\right\}}{\left\{1-e^{\frac{2n\omega_2 i\pi}{\omega_1}}\right\}^2}$$

$$=\frac{1-2q^{2n}\cos\dfrac{\pi z}{\omega_1}+q^{4n}}{(1-q^{2n})^2}.$$

Now if the imaginary part of ω_2/ω_1 is positive, we have $|q|<1$; and thus the infinite product

$$\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos\dfrac{\pi z}{\omega_1}+q^{4n}}{(1-q^{2n})^2}$$

converges absolutely, since the series

$$\sum_{n=1}^{\infty} q^{2n}$$

converges absolutely; and hence we can separate off the exponential factors, and can write

$$\sigma(z)=e^{Cz^2}\frac{2\omega_1}{\pi}\sin\frac{\pi z}{2\omega_1}\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos\dfrac{\pi z}{\omega_1}+q^{4n}}{(1-q^{2n})^2},$$

where C is a constant.

The quantity C can be very simply determined from the relation

$$\sigma(z+2\omega_1)=-e^{2\eta_1(z+\omega_1)}\sigma(z);$$

for this gives

$$e^{C(z+2\omega_1)^2}=e^{Cz^2+2\eta_1(z+\omega_1)},$$

or

$$C=\frac{\eta_1}{2\omega_1}.$$

We have therefore finally an expression for $\sigma(z)$ as a singly-infinite product, namely

$$\sigma(z) = e^{\frac{\eta_1 z^2}{2\omega_1}} \frac{2\omega_1}{\pi} \sin\frac{\pi z}{2\omega_1} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n}\cos\frac{\pi z}{\omega_1} + q^{4n}}{(1-q^{2n})^2},$$

where $q = e^{\frac{i\pi\omega_2}{\omega_1}}$.

214. *The integration of elliptic functions.*

The integral of any elliptic function can be found in terms of the functions $\zeta(z)$ and $\sigma(z)$, by using the theorem given in § 211, on the resolution of elliptic functions into a sum of ζ-functions.

In fact, in § 211 an expression

$$c + \sum_{k=1}^{n} \sum_{s=1}^{r_k} \frac{(-1)^{s-1}}{(s-1)!} c_{k_s} \zeta^{(s-1)}(z - a_k)$$

has been found for the elliptic function $f(z)$; the indefinite integral of this expression is

$$cz + \sum_{k=1}^{n} c_{k_1} \log\sigma(z-a_k) + \sum_{k=1}^{n} \sum_{s=2}^{r_k} \frac{(-1)^{s-1}}{(s-1)!} c_{k_s} \zeta^{(s-2)}(z-a_k),$$

which is the required integral of $f(z)$.

Example. The expression for $\wp^2(z)$, found by the theorem of § 211, is

$$\tfrac{1}{6}\wp''(z) + \tfrac{1}{12}g_2.$$

It follows that

$$\int \wp^2(z)\,dz = \tfrac{1}{6}\wp'(z) + \tfrac{1}{12}g_2 z + \text{Constant}.$$

215. *Expression of an elliptic function whose zeros and poles are known.*

We have already seen (§ 205) that the number of irreducible zeros of an elliptic function is equal to the number of its irreducible poles; and that (§ 208) the sum of the affixes of the zeros differs from the sum of the affixes of the poles only by a quantity of the form $(2m\omega_1 + 2n\omega_2)$, where m and n are integers. By replacing the zeros and poles by others congruent to them, we can reduce this difference to zero. Suppose this done, so that for a given function $f(z)$ the irreducible zeros are $a_1, a_2, \ldots a_n$, and the irreducible poles are $b_1, b_2, \ldots b_n$, where

$$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n.$$

If any of the zeros or poles is multiple, of order k say, it will of course be counted as if it were k distinct simple zeros or poles.

Consider now the quantity

$$E(z) = \frac{\sigma(z-a_1)\,\sigma(z-a_2) \ldots\ldots \sigma(z-a_n)}{\sigma(z-b_1)\,\sigma(z-b_2) \ldots\ldots \sigma(z-b_n)}.$$

We have

$$E(z+2\omega_1) = e^{2\eta_1\{(z-a_1)+(z-a_2)+\ldots+(z-a_n)-(z-b_1)-\ldots-(z-b_n)\}}E(z)$$
$$= E(z).$$

Similarly $$E(z+2\omega_2) = E(z).$$

Thus $E(z)$ is an elliptic function, with the same periods as $f(z)$; and therefore $f(z)/E(z)$ admits these periods.

But the function $f(z)/E(z)$ clearly has no zeros or poles at the points

$$a_1,\ a_2,\ \ldots\ a_n,\ b_1,\ \ldots\ b_n,$$

and so has no zeros or poles at any point of the z-plane. Therefore, by Liouville's theorem, $f(z)/E(z)$ is a constant; and so finally

$$f(z) = c\frac{\sigma(z-a_1)\,\sigma(z-a_2)\ \ldots\ldots\ \sigma(z-a_n)}{\sigma(z-b_1)\,\sigma(z-b_2)\ \ldots\ldots\ \sigma(z-b_n)},$$

where c is some constant.

An elliptic function is therefore determinate, save for a multiplicative constant, when the places of its irreducible zeros and poles are known.

This is analogous to the factorisation of a rational function: if a rational function has zeros at points $a_1, a_2, \ldots a_n$, and poles at points $b_1, b_2, \ldots b_n$, it can be expressed in the form

$$c\frac{(z-a_1)(z-a_2)\ldots(z-a_n)}{(z-b_1)(z-b_2)\ldots(z-b_n)},$$

where c is a constant.

Example 1. Prove that

$$\wp(z)-\wp(y) = -\frac{\sigma(z+y)\,\sigma(z-y)}{\sigma^2(z)\,\sigma^2(y)}.$$

By differentiating this formula, shew that

$$\frac{1}{2}\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)} = \zeta(z+y)-\zeta(z)-\zeta(y),$$

and by further differentiation obtain the addition-theorem

$$\wp(z+y) = -\wp(z)-\wp(y)+\frac{1}{4}\left\{\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\right\}^2.$$

Example 2. If

$$\sum_{\lambda=1}^{n}(a_\lambda-b_\lambda)=0,$$

shew that $$\sum_{\lambda=1}^{n}\frac{\sigma(a_\lambda-b_1)\ldots\sigma(a_\lambda-b_\lambda)\ldots\sigma(a_\lambda-b_n)}{\sigma(a_\lambda-a_1)\ldots 1\ldots\sigma(a_\lambda-a_n)}=0.$$

Miscellaneous Examples.

1. Shew that, if p denote one of the functions $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$, and if q and r denote the other two, it is always possible to choose constants a, b, c, such that

$$\int^z p\,dz = a \log (bq + cr).$$

2. Shew that every elliptic function of order n can be expressed as the quotient of two expressions of the form

$$a_1 \wp (z+b) + a_2 \wp' (z+b) + \ldots + a_n \wp^{(n-1)} (z+b),$$

where b, a_1, a_2, ... a_n, are constants. (Painlevé.)

3. Prove that

$$\begin{aligned}\wp (z-a)\, \wp (z-b) = \wp (a-b) \{\wp (z-a) + \wp (z-b) - \wp (a) - \wp (b)\} \\ + \wp' (a-b) \{\zeta (z-a) - \zeta (z-b) + \zeta (a) - \zeta (b)\} \\ + \wp (a)\, \wp (b).\end{aligned}$$

(Cambridge Mathematical Tripos, Part II, 1895.)

4. Shew that

$$\frac{\sigma (x+y+z)\, \sigma (x-y)\, \sigma (y-z)\, \sigma (z-x)}{\sigma^3 (x)\, \sigma^3 (y)\, \sigma^3 (z)} = -\frac{1}{2} \begin{vmatrix} 1 & \wp (x) & \wp' (x) \\ 1 & \wp (y) & \wp' (y) \\ 1 & \wp (z) & \wp' (z) \end{vmatrix}.$$

Obtain the addition-theorem for the function $\wp (z)$ from this result.

5. Establish the identity

$$\begin{vmatrix} 1 & \wp (z_0) & \wp' (z_0) \ldots \wp^{(n-1)} (z_0) \\ 1 & \wp (z_1) & \wp' (z_1) \ldots \wp^{(n-1)} (z_1) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 1 & \wp (z_n) & \wp' (z_n) \ldots \wp^{(n-1)} (z_n) \end{vmatrix} = (-1)^{\frac{1}{2}n(n-1)}\, 1\,!\, 2\,! \ldots n\,!\, \frac{\sigma (z_0 + z_1 + \ldots + z_n)\, \Pi \sigma (z_\lambda - z_\mu)}{\sigma^{n+1} (z_0) \ldots \sigma^{n+1} (z_n)},$$

where the product is taken for all integer values of λ and μ from 0 to n, with the restriction $\lambda < \mu$.

6. Prove that

$$\zeta (z-a) - \zeta (z-b) - \zeta (a-b) + \zeta (2a-2b) = \frac{\sigma (z-2a+b)\, \sigma (z-2b+a)}{\sigma (2b-2a)\, \sigma (z-a)\, \sigma (z-b)}.$$

(Cambridge Mathematical Tripos, Part II, 1895.)

7. Shew that, if $z_0 + z_1 + z_2 + z_3 = 0$, then

$$\{\Sigma \zeta (z_\lambda)\}^3 = 3 \{\Sigma \zeta (z_\lambda)\} \{\Sigma \wp (z_\lambda)\} + \Sigma \wp' (z_\lambda),$$

the summations being taken for $\lambda = 0, 1, 2, 3$.

(Cambridge Mathematical Tripos, Part II, 1897.)

8. Prove that

$$g(z) = \frac{\sigma(z+z_1)\,\sigma(z+z_2)\,\sigma(z+z_3)\,\sigma(z+z_4)}{\sigma\{2z+\frac{1}{2}(z_1+z_2+z_3+z_4)\}}$$

is a doubly-periodic function of z, such that

$$g(z)+g(z+\omega_1)+g(z+\omega_2)+g(z+\omega_1+\omega_2)$$
$$= -2\sigma\{\tfrac{1}{2}(z_2+z_3-z_1-z_4)\}\;\sigma\{\tfrac{1}{2}(z_3+z_1-z_2-z_4)\}\;\sigma\{\tfrac{1}{2}(z_1+z_2-z_3-z_4)\}.$$

(Cambridge Mathematical Tripos, Part II, 1893.)

9. If $f(z)$ be a doubly-periodic function of the third order, with poles at $z=c_1$, $z=c_2$, $z=c_3$, and if $\phi(z)$ be a doubly-periodic function of the second order with the same periods and poles at $z=a$, $z=\beta$, its value in the neighbourhood of $z=a$ being

$$\phi(z) = \frac{\lambda}{z-a} + \lambda_1(z-a) + \lambda_2(z-a)^2 + \ldots,$$

prove that

$$\tfrac{1}{2}\lambda^2\{f''(a)-f''(\beta)\} - \lambda\{f'(a)+f'(\beta)\}\sum_1^3 \phi(c_1) + \{f(a)-f(\beta)\}\{3\lambda\lambda_1 + \sum_1^3 \phi(c_2)\,\phi(c_3)\} = 0.$$

(Cambridge Mathematical Tripos, Part II, 1894.)

10. If $\lambda(z)$ be an elliptic function with two poles a_1, a_2, and if $z_1, z_2, \ldots z_{2n}$, be $2n$ arbitrary arguments such that

$$z_1+z_2+\ldots+z_{2n} = n(a_1+a_2),$$

shew that the determinant whose nth row is

$$1,\ \lambda(z_i),\ \lambda^2(z_i),\ \ldots\ \lambda^n(z_i),\ \lambda_1(z_i),\ \lambda(z_i)\,\lambda_1(z_i),\ \lambda^2(z_i)\,\lambda_1(z_i),\ \ldots\ \lambda^{n-2}(z_i)\,\lambda_1(z_i),$$

where
$$\lambda_1(z_i) = \frac{d}{dz_i}\lambda(z_i),$$
vanishes identically.

(Cambridge Mathematical Tripos, Part II, 1893.)

11. Shew that, provided certain conditions of inequality are satisfied,

$$\frac{\sigma(z+y)}{\sigma(z)\,\sigma(y)}\,e^{\frac{-\eta_1 zy}{\omega_1}} = \frac{\pi}{2\omega_1}\left(\cot\frac{\pi z}{2\omega_1} + \cot\frac{\pi y}{2\omega_1}\right) + \frac{2\pi}{\omega_1}\Sigma q^{2mn}\sin\frac{\pi}{\omega_1}(mz+ny),$$

where the summation applies to all positive integer values of m and n.

(Cambridge Mathematical Tripos, Part II, 1895.)

12. Assuming the formula

$$\sigma(z) = e^{\frac{\eta_1 z^2}{2\omega_1}}\cdot\frac{2\omega_1}{\pi}\sin\frac{\pi z}{2\omega_1}\prod_1^\infty \frac{1-2q^{2s}\cos\dfrac{\pi z}{\omega_1}+q^{4s}}{(1-q^{2s})^2},$$

prove that

$$\wp(z) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2\operatorname{cosec}^2\frac{\pi z}{2\omega_1} - 2\left(\frac{\pi}{\omega_1}\right)^2\sum_1^\infty \frac{sq^{2s}}{1-q^{2s}}\cos\frac{s\pi z}{\omega},$$

on condition that

$$-2R\left(\frac{\omega_2}{i\omega_1}\right) < R\left(\frac{z}{i\omega_1}\right) < 2R\left(\frac{\omega_2}{i\omega_1}\right).$$

(Cambridge Mathematical Tripos, Part II, 1896.)

13. Shew that

$$\int \{(x^2-a)(x^2-b)\}^{-\frac{1}{2}}\,dx = -\frac{1}{2}\log\frac{\sigma(z-z_0)}{\sigma(z+z_0)} + \frac{i}{2}\log\frac{\sigma(z-iz_0)}{\sigma(z+iz_0)},$$

where

$$x^2 = a + \frac{1}{6}\,\frac{1}{\wp^2(z) - \wp^2(z_0)},$$

$$g_2 = \frac{2b}{3a(a-b)},$$

$$g_3 = 0,$$

$$\wp^2(z_0) = \frac{1}{6(a-b)}.$$

(Dolbnia.)

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